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Conflicts in regular networks[☆]

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ABSTRACT

This paper considers a regular conflict network model with the returns to scale technology. Agents are asymmetric in terms of their effort costs. We show that the impact of the returns to scale technology on agents' behaviors crucially depends on the cost asymmetry among agents. When the cost asymmetry is sufficiently high, both individual total efforts and the conflict intensity can have an inverted U relationship with the level of returns to scale.

1. Introduction

Conflicts in social networks are widely observed in real-life situations, which range from sports, warfare, trade negotiations to firm competitions in various markets. To capture the strategic interactions among parties in these conflict environments, the growing literature has developed conflict network models by embedding multi-battle contests into network structures.[1] Unlike traditional contest models which typically ignore the interdependencies between different conflicts, these models are concerning the interrelationships between different conflicts.

Among all possible network structures, a regular network is perhaps the most classical one in which each agent is involved in the same number of battles. For instance, in various intergovernmental organizations (e.g., World Trade Organization, World Health Organization), each country is expected to negotiate bilaterally with all other countries in international affairs negotiations. Similarly, each firm is simultaneously competing with all other rivals in different product markets or geographical areas. In such regular conflict environments, there is no doubt that the conflict technology plays an important role in influencing the strategic interactions among the agents. Jiao et al. (2019) formally model the conflict technology using the degree of returns to scale in a general Tullock contest model. By focusing on the complete bipartite networks, they have made the first attempt to address how the conflict technology may affect agents' behaviors. In this study, we investigate this question further in regular networks, where each agent participates in the same number of bilateral conflicts.

In real-life situations, it is possible for some agents to have different capabilities than others. In international trade negotiations, some country has a stronger bargaining power compared to others. In business competition, when a new entrant enters a market which has been occupied by several dominant firms, it may struggle to survive. Such observations naturally raise the following questions: Given the ability asymmetry, what is the relationship between total efforts of different agents and the conflict technology? Moreover, how does this relation depend on the degree of asymmetry among agents?

To address these questions, we set up a regular conflict network model with the returns to scale technology. There is a singular agent, whose ability is different from those of others in terms of effort cost. Those remaining agents who have the same abilities are referred to as the common agents. We first characterize a unique pure-strategy Nash equilibrium, and further analyze

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[1] See, for example, Franke and Özuürk (2015), König et al. (2017) and Kimbrough et al. (2020).

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how the conflict technology affects the equilibrium behavior of each agent.

Our key finding is that the effect of returns to scale technology on agents' equilibrium behaviors depends critically on the degree of asymmetry among agents. As the returns to scale technology increases, the total effort of each common agent always increases, but the total effort of the singular agent may change non-monotonically. In particular, when agents differ sufficiently in cost asymmetry, there exists an inverted U relationship between the total effort of the singular agent and the level of returns to scale. We further establish a similar inverted U relationship between the conflict intensity and the returns to scale technology, when the singular agent is extremely strong.

The intuition behind the above results can be explained by a competition effect, a discouragement effect and a substitution effect. A higher level of returns to scale intensifies competition among all agents, therefore causing them to exert more effort (competition effect). The weaker agent in a bilateral conflict is less likely to win as the level of returns to scale increases, and hence she is discouraged from investing further and tends to lower her own effort level (discouragement effect). Since each common agent has to compete with many identical rivals, she has an incentive to increase the efforts in those conflicts and relatively decrease the effort in the conflict against the singular agent (substitution effect). For the singular agent, the competition effect is dominated by the discouragement effect when the cost asymmetry is sufficiently high, which results in an inverted U relationship between her individual total effort and the returns to scale technology. For a common agent, although the substitution effect is ambiguous, the aggregate effect always raises her total effort as the level of returns to scale increases. Furthermore, since the total effort of the stronger agent(s) always accounts for a large percentage of the conflict intensity, the conflict intensity tends to have a similar shape with the individual total effort of the stronger agent(s).

There is a burgeoning literature that studies conflicts among agents using network models. Franke and Özuürk (2015) develop a model of conflict networks, in which players are involved in several bilateral conflicts and each bilateral conflict is a lottery contest (i.e., $r = 1$). They examine how the conflict intensity can be affected by the network structure. Jiao et al. (2019) extend their framework by allowing for a general Tullock contest with the returns to scale technology. They investigate the relationship between the conflict intensity and the returns to scale technology in bipartite conflict networks.[2] Xu et al. (2022) consider a more general conflict network model and characterize the set of pure-strategy equilibria. Bozbay and Vesperoni (2018) axiomatically characterize a contest success function for networks. Kovenock and Roberson (2018) investigate the attack and defense of multiple networks of targets with intra network strategic complementarity among targets. Dziubiński et al. (2021) propose a dynamic model of conflict networks and show that the dynamics of conflicts are shaped by factors including the technology of war, resources and contiguity network. Cortes-Corrales and Gorny (2022) study how changing the strength of other symmetric agents induces knock-on effects throughout a multi-sided-weighted network of conflicts.

Our work is also related to the contest literature that studies the effect of the returns to scale technology on equilibrium efforts. A handful of studies allow the dissipation factor to be a primary instrument of the contest designer.[3] Nti (2004) analyzes the asymmetric valuations situation and determines the optimal contest technology for different profiles of player valuations. In a general Tullock contest with two asymmetric players, Wang (2010) shows that the contest designer adopts a lower level of discriminatory power r that maximizes the aggregate effort, as the players become more heterogeneous. For contests with endogenous entry, Fu et al. (2015) show that an increase in the discriminatory power r may result in fewer entrants. Nevertheless, there also exists an interior optimum level of discriminatory power, which maximizes the aggregate effort. Letina et al. (2023) study the optimal design of the contest success function, and show that the effort-maximizing contest will typically feature an intermediate level of discriminatory power. Unlike those works, which consider the optimal design problem, Feng and Lu (2018) study how the optimal prize allocation in a sequential three-battle contest varies with the discriminatory power of the contest technology. More recently, Fu and Wu (2022) and Lu et al. (2017, 2022) consider optimal design problem for a wide range of accuracy level of the contest. Unlike most of the optimal design problems that assume contest designers can choose a mechanism to maximize effort, our paper treats the dissipation factor as an exogenous value, which cannot be altered by any individual conflict party. Additionally, we investigate the effect of the dissipation factor on agents' behaviors.

The remainder of the paper is organized as follows. In Section 2, we formally introduce the regular conflict network model and provide the equilibrium analysis. In Section 3, we study the effects of the returns to scale technology on equilibrium efforts of agents. Section 4 concludes. All technical proofs are relegated in Appendix.

2. Model and equilibrium

Consider a regular conflict network among $n + 1$ agents, in which each agent is in a bilateral conflict with all other agents. The set of agents is denoted by $\mathcal{N} = \{1, 2, \ldots, n + 1\}$. We model each bilateral conflict among agents as a contest. The outcome of each contest depends on the simultaneous strategic behaviors (i.e., contest efforts) of the involved agents. For each agent i, denote her effort in the contest against her rival j by $x_{ij} \in \mathbb{R}_+$ (we use $\mathbb{R}_+$ to denote the set of all nonnegative real numbers), her effort vector against all of her rivals by an n-dimensional vector $\boldsymbol{x}_i = (x_{ij})_{j\neq i}$, and her total effort by $X_i = \sum_{j\neq i} x_{ij}$.

In each contest, the winning agent obtains an exogenous prize $V > 0$ and the agent who loses receives nothing. The outcome of each contest is specified by the Tullock contest success function: In the contest between agent i and her rival j, when they exert efforts x_{ij} and x_{ji} respectively, the winning probability of agent i is

$$p(x_{ij}, x_{ji}) = \begin{cases} \frac{x_{ij}^r}{x_{ij}^r + x_{ji}^r}, & \text{if } x_{ij} + x_{ji} > 0, \\ \frac{1}{2}, & \text{if } x_{ij} = x_{ji} = 0, \end{cases}$$

[2] Both this paper and Jiao et al. (2019) find an inverted U relationship between equilibrium efforts and the returns to scale technology. They explore the same question from different perspectives. Unlike (Jiao et al., 2019), which analyzes a complete bipartite conflict network with structure asymmetry, this paper studies a symmetric network structure with the heterogeneity of agents. In Jiao et al. (2019), as long as the structure asymmetry is sufficiently large, the individual total effort of each agent has an inverted U relationship with the returns to scale technology. In this paper, we find that, regardless of how large the cost asymmetry is, the total effort of some agents does not show such an inverted U relationship. Furthermore, Jiao et al. (2019) show that the conflict intensity has an inverted U relationship with the returns to scale technology when the structure asymmetry is sufficiently large. While we find that the conflict intensity may be still increasing in the level of returns to scale even when the cost asymmetry is sufficiently large.

[3] In the contest literature, the terminologies "the return to scale technology" and "dissipation factor" are used in different situations. They both refer to the discriminatory power in the Tullock contest.

where r is a positive constant and represents the *returns to scale technology* in effort spending. Throughout the paper, we assume that $r \leq 1$, which ensures the existence and uniqueness of a pure-strategy Nash equilibrium.[4]

We model the asymmetry among agents by assuming that they have different costs. Specifically, the cost function of agent 1 is $C_1(X_1) = \frac{\beta}{2}X_1^2$, and that of other agents $j \neq 1$ is $C_j(X_j) = \frac{1}{2}X_j^2$, where β is a positive constant. Notice that agent 1 is a strong agent when $0 < \beta < 1$ and a weak one when $\beta > 1$. For convenience, agent 1 is usually referred to as the *singular agent*, and other agents are referred to as *common agents*.

When the agents' strategy profile is $\boldsymbol{x} = (\boldsymbol{x}_i)_{i\in\mathcal{N}}$, the expected payoff function of each agent i is

$$u_i(\boldsymbol{x}) = V \cdot \sum_{j\neq i} p(x_{ij}, x_{ji}) - C_i(X_i) = V \cdot \sum_{j\neq i} p(x_{ij}, x_{ji}) - C_i\big(\textstyle\sum_{j\neq i} x_{ij}\big).$$

By Theorem 2 and Lemma 2 in Xu et al. (2022), this conflict network game admits a unique pure-strategy Nash equilibrium, which is interior and symmetric. The following proposition formally characterizes this equilibrium.

Proposition 1 (*Equilibrium*)**.** *There exists a unique pure-strategy Nash equilibrium, in which the equilibrium effort of the singular agent against each rival is a, the equilibrium effort of each common agent against the singular agent is b, and the equilibrium effort of each common agent against another common agent is c:*

$$a = \left[\frac{(n-1)rV\theta^2}{4n\beta(n\beta-\theta^2)}\right]^{\frac{1}{2}}, \quad b = \left[\frac{(n-1)rV\theta^4}{4n\beta(n\beta-\theta^2)}\right]^{\frac{1}{2}}, \quad \textit{and}$$

$$c = \left[\frac{rV(n\beta-\theta^2)}{4n(n-1)\beta}\right]^{\frac{1}{2}};$$

Here $\theta := \frac{b}{a} \in \big(0, \sqrt{n\beta}\big)$ *is uniquely determined by the following equation*

$$4(n\beta-\theta^2)\theta^{r-2} - (n-1)(1+\theta^r)^2 = 0. \tag{1}$$

Fig. 1 explicitly depicts a regular conflict network with four players and the equilibrium efforts in each bilateral conflict.

The parameter $\theta = \frac{b}{a}$ in Proposition 1 captures the ratio of equilibrium efforts in any bilateral conflict between a common agent and the singular agent. For each $\beta > 0$ and $r \in (0, 1]$, we can determine the unique solution θ of Eq. (1). Such a solution θ depends on β and r, which will be rewritten as $\theta(\beta, r)$ to explicitly reflect those dependence. We define an auxiliary function

$$\delta(t) = \frac{(n-1)t^3 + 2(n+1)t^2 + (n-1)t}{4n}.$$

In the following, we consider the relationship between the equilibrium ratio θ and the cost parameter β.

Lemma 1 (*Effect of β on θ*)**.** *The relationship between $\theta = \theta(\beta, r)$ and β is summarized as follows:*

(1) For any $r \in (0, 1]$, the equilibrium effort ratio θ is increasing in β. Moreover, we have $0 < \theta < 1$ when $0 < \beta < 1$, $\theta = 1$ when $\beta = 1$, and $\theta > 1$ when $\beta > 1$.

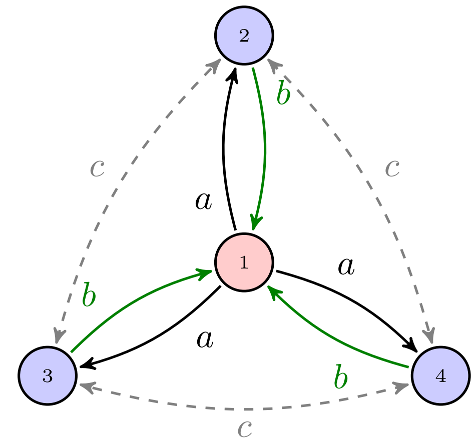


Fig. 1. A regular conflict network and its equilibrium efforts.

(2) When $\beta \neq 1$, the equilibrium effort ratio θ lies in the interval $(\underline{\theta}, \beta^{\frac{1}{2}})$, where $\underline{\theta}$ is the unique positive solution to $\delta(\theta) = \beta$.[5]

In each conflict between the singular agent and a common agent, Lemma 1 shows that as the singular agent becomes weaker (or β increases), this equilibrium effort ratio θ will increase. This is in line with the conventional wisdom that agents will exert less (resp. more) effort when they believe they have a smaller (resp. larger) chance of winning.

3. Comparative statics

In this section, we study the effect of r on the equilibrium efforts, including (1) the effort of each agent against each opponent, (2) the individual total effort of each agent, and (3) the conflict intensity (i.e., the total effort of all agents).

Before the analysis, we first prove the following auxiliary result.

Lemma 2 (*Effect of r on θ*)**.** *For any $\beta \neq 1$, the equilibrium effort ratio $\theta = \frac{b}{a}$ is decreasing in r. Moreover, $\lim_{r\downarrow 0}\theta(\beta, r) = \beta^{\frac{1}{2}}$ and $\lim_{r\uparrow 1}\theta(\beta, r) = \underline{\theta}$, where $\underline{\theta}$ is the unique positive solution to $\delta(\theta) = \beta$.*

Lemma 2 suggests that the equilibrium effort ratio of the common agents to the singular agent decreases with the level of returns to scale. As the level of returns to scale r increases, the marginal benefit of exerting effort in each bilateral conflict becomes higher, which tends to incentivize each agent to exert more effort. However, the asymmetry among agents makes their incentives to supply effort diverge. By raising the effort level b, a common agent can only get some benefit in a single bilateral conflict with the singular agent. By contrast, the singular agent can benefit more by raising the effort level a, as she is involved in n bilateral conflicts with all common agents. Hence, the singular agent has a stronger incentive to raise his effort supply than any common agent does, which explains why the effort ratio θ is decreasing in r.[6] Additionally, this result does not reveal how conflict technology impacts the equilibrium behaviors of each agent. We will address this in the next subsection.

[4] In a Tullock contest, the parameter r measures the discriminatory power or the noisiness of the contest. A higher r means that the contest is more discriminatory or less noisy, and hence an increase in effort will result in a higher rate of return for a particular agent. In particular, if $r < 1$ (resp. $r = 1$ or $r > 1$), the returns to scale are decreasing (resp. constant or increasing). In other words, when $r < 1$, although the marginal benefit is always positive, it is decreasing as the effort level increases.

[5] When $0 < \beta < 1$, the equilibrium effort ratio θ is in the interval $(\beta, \beta^{\frac{1}{2}})$; when $\beta > 1$, the equilibrium effort ratio θ is in $(\beta^{\frac{1}{3}}, \beta^{\frac{1}{2}})$.

[6] This can also be explained by the substitution effect describing below.

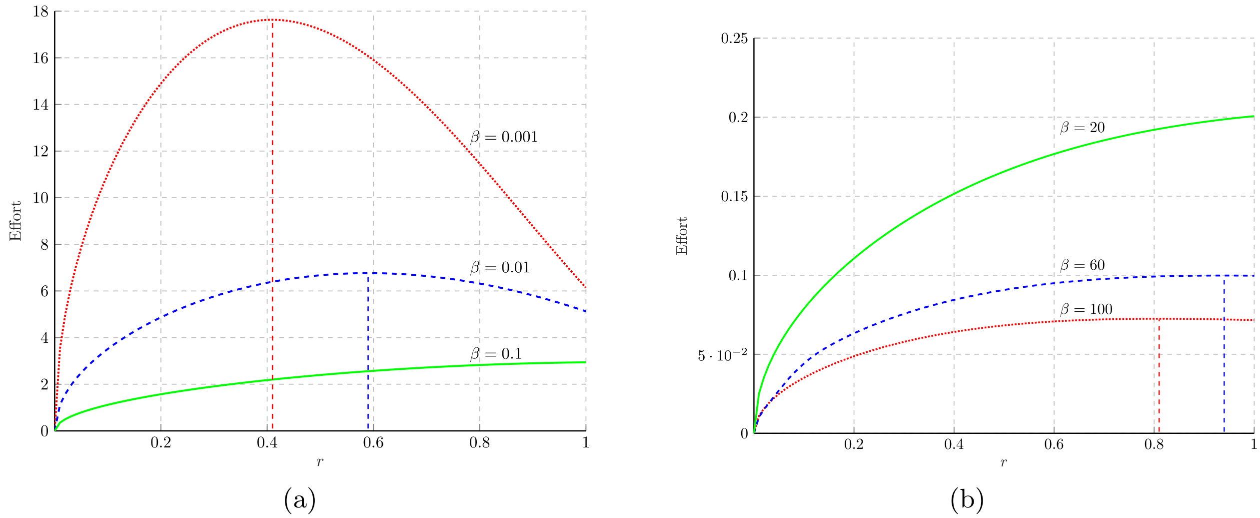


Fig. 2. The relationship between equilibrium effort a and returns to scale technology r.

3.1. Effect of r on a, b and c

We first consider the effect of r on a. The result is summarized in the following proposition.

Proposition 2 (*Effect of r on a*). *There exist $\underline{\beta}_a < 1$ and $\bar{\beta}_a > 1$ such that*

- *when $\underline{\beta}_a \le \beta \le \bar{\beta}_a$, the equilibrium effort a is strictly increasing in r.*
- *when $\beta < \underline{\beta}_a$ or when $\beta > \bar{\beta}_a$, there exists an inverted U relationship between the equilibrium effort a and r. In particular, there exists $r_a \in (0, 1)$ such that a is strictly increasing in r when $r \in (0, r_a)$ and strictly decreasing in r when $r \in (r_a, 1]$.*

To see how the relationship between the equilibrium effort a and the returns to scale technology r is affected by the cost asymmetry β, we consider an example in Fig. 2. Given $V = 10$ and $n = 2$, Fig. 2 depicts the equilibrium efforts of the singular agent with β being 0.001, 0.01, 0.1, 20, 60, and 100. Indeed, we have $\underline{\beta}_a = 0.08$ and $\bar{\beta}_a = 49.63$. So, when β is 0.1 or 20 (between 0.08 and 49.63), the equilibrium effort a is strictly increasing in r. And when β is any other value (outside $(0.08, 49.63)$), there exists an inverted U relationship between the equilibrium effort a and r.

We then consider the effect of r on b. The result is summarized in the following proposition.

Proposition 3 (*Effect of r on b*). *There exist $\underline{\beta}_b < 1$ and $\bar{\beta}_b > 1$ such that*

- *when $\underline{\beta}_b \le \beta \le \bar{\beta}_b$, the equilibrium effort b is increasing in r.*
- *when $\beta < \underline{\beta}_b$ or when $\beta > \bar{\beta}_b$, there exists an inverted U relationship between the equilibrium effort b and r. In particular, there exists $r_b \in (0, 1)$ such that b is strictly increasing in r when $r \in (0, r_b)$ and strictly decreasing in r when $r \in (r_b, 1]$.*

Given $V = 10$ and $n = 2$, Fig. 3 depicts the equilibrium efforts of the singular agent with β being 0.01, 0.1, 0.5, 5, 20, and 60. Indeed, we have $\underline{\beta}_b = 0.13$ and $\bar{\beta}_b = 16.33$. So, when β is 0.5 or 5 (between 0.13 and 16.33), the equilibrium effort b is strictly increasing in r. And when β is any other value (outside $(0.13, 16.33)$), there exists an inverted U relationship between the equilibrium effort b and r.

The following result studies the relationships between the effect of r on a and the effect of r on b.

Lemma 3. *We have the following relationships.*

- *When a is decreasing in r, b must also be decreasing in r.*
- *When $\beta < \underline{\beta}_a = \min\{\underline{\beta}_a, \underline{\beta}_b\}$ or $\beta > \bar{\beta}_a = \max\{\bar{\beta}_a, \bar{\beta}_b\}$, we have $r_a > r_b$.*

The first result clearly indicates that b is more likely to form an inverted U relationship between r than a, which implies the second result. When both a and b have inverted U relationships with r, it should be the case that $r_a > r_b$. Otherwise, in some interval r, a is increasing but b is decreasing.[7]

Propositions 2 and 3 demonstrate that the effects of the returns to scale technology on efforts in a single bilateral conflict between the singular agent and a common agent (a and b) crucially depend on the cost asymmetry among agents. When agents are not sufficiently asymmetric, those efforts are always increasing in r. However, when agents are sufficiently asymmetric, there exists an inverted U relationship between those efforts and the level of returns to scale.

Intuitively, there are three effects: a competition effect, a discouragement effect and a substitution effect. First, as the level of returns to scale increases, the competition between agents in each bilateral conflict becomes fiercer, implying that each agent has a greater incentive to exert effort (competition effect). Second, in a bilateral conflict between a singular agent and a common agent, the weaker agent is less likely to win as r increases, and hence she is discouraged from investing further and wants to lower her own effort level (discouragement effect). The discouragement effect can be captured by $\theta = \frac{b}{a}$. When $\beta < 1$, as r increases, $\theta < 1$ moves further away from 1, and hence the discouragement effect becomes stronger. When $\beta > 1$, as r increases, $\theta > 1$ moves close to 1, and hence the discouragement effect becomes weaker. Third, since each common agent has to compete with a singular agent and $n-1$ identical rivals, common agents have incentives to increase c and relatively decrease b if the cost asymmetry exists (substitution effect). Such an effect becomes stronger as r increases.[8]

Notice that all the three effects can influence b, while only the first two effects have impacts on a. Since the competition effect impacts individual efforts a and b in the opposite direction

[7] This can be explained by the substitution effect describing below.

[8] The substitution effect can be captured by the effort ratio $\rho = \frac{b}{c} = \frac{(n-1)\theta^2}{n\beta - \theta^2}$, which is decreasing in r.

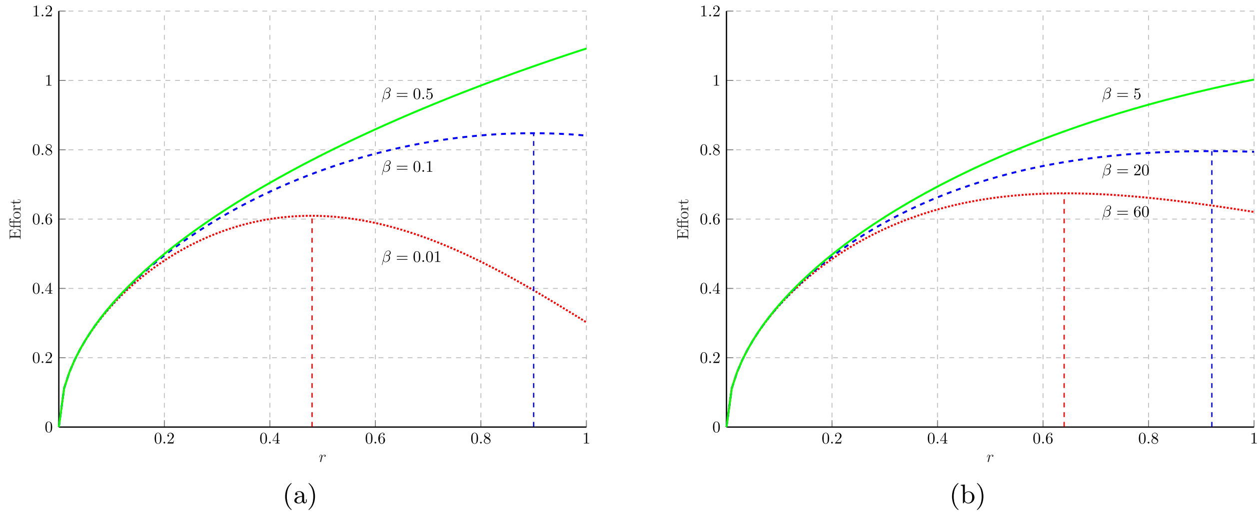


Fig. 3. The relationship between equilibrium effort b and returns to scale technology r.

as compared to other two effects, the net effect of the increase in returns to scale technology on these individual efforts would then depend on whether the competition effect dominates. When the cost asymmetry is sufficiently low, the competition effect always dominates the discouragement effect for the singular agent, and it also dominates both the discouragement effect and the substitution effect for each common agent. Hence, the individual efforts a and b always increase as the level of returns to scale increases. When the cost asymmetry is moderate, the competition effect continues to dominate for the singular agent, which implies an increasing relationship between a and r. However, it can only dominate the discouragement effect and the substitution effect for each common agent when r is sufficiently small, which implies that there exists an inverted U relationship between b and r. When the cost asymmetry is sufficiently high, the competition effect only dominates when r is sufficiently small for both types of agents, which results in an inverted U relationship between a, b and r.

From the numerical examples in Figs. 2 and 3, we can see how critical values r_a and r_b change as β changes. Our observation suggests: (1) When $\beta < \underline{\beta}_a$ (resp. $\underline{\beta}_b$), r_a (resp. r_b) increases as β increases; (2) When $\beta > \bar{\beta}_a$ (resp. $\bar{\beta}_b$), r_a (resp. r_b) decreases as β increases. It means that the critical levels of return to scale (r_a and r_b) increase in the cost asymmetry β when the singular agent is sufficiently stronger than the common agents, while decrease in the cost asymmetry when the singular agent is sufficiently weaker than the common agents. Intuitively, as the discouragement effect and substitution effect increase in the cost asymmetry β, it is more likely to dominate the competition effect as the differences between the singular agent and the common agents become more pronounced. Consequently, the greater the ability difference, the sooner the critical value appears.

We last consider the effect of r on c.

Proposition 4 (*Effect of r on c*)**.** *The equilibrium effort c is strictly increasing in r regardless of β.*

This proposition simply means that as the level of returns to scale r increases, the conflict becomes more intensive so that it tends to elicit higher individual effort among the common agents. It is the outcome of the combination of the competition effect and (positive) substitution effect.

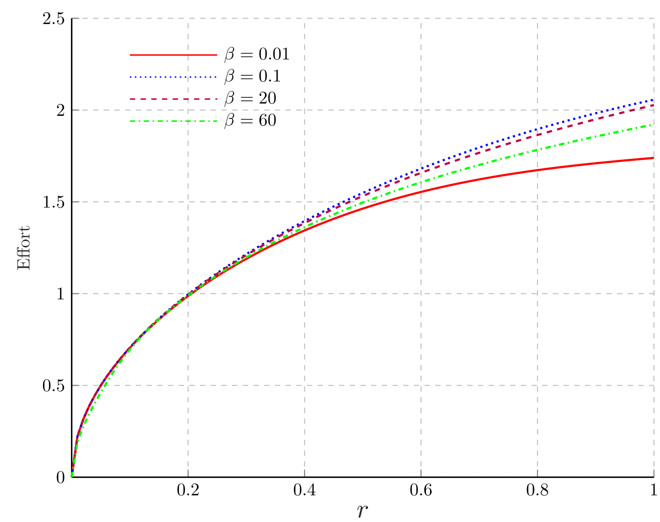


Fig. 4. The relationship between individual total effort $b + (n-1)c$ and r.

3.2. Individual total efforts and conflict intensity

It is clear that the singular agent's total effort na has the same shape with a as r increases. We further explore how does the returns to scale technology affect the individual total efforts of each common agent.

Proposition 5 (*Effect of r on Total Effort of Common Agents*)**.** *The individual total effort $b + (n-1)c$ of any common agent is strictly increasing in r, regardless of β.*

Given $V = 10$ and $n = 2$, Fig. 4 depicts the individual total efforts of each common agent with β being 0.1, 0.01, 20, and 60.

Since each common agent has to compete with a singular agent and $n-1$ identical companions, the total effort of each common agent is $b + (n-1)c$. Note that c is strictly increasing in r while b may have an inverted U relationship with r, the total effect is not straightforward. Due to the intensive competitions among the common agents, the competition effect strictly dominates the substitution effect and the potential discouragement effect. Thus, the individual total effort of each common agent always increases with the level of returns to scale.

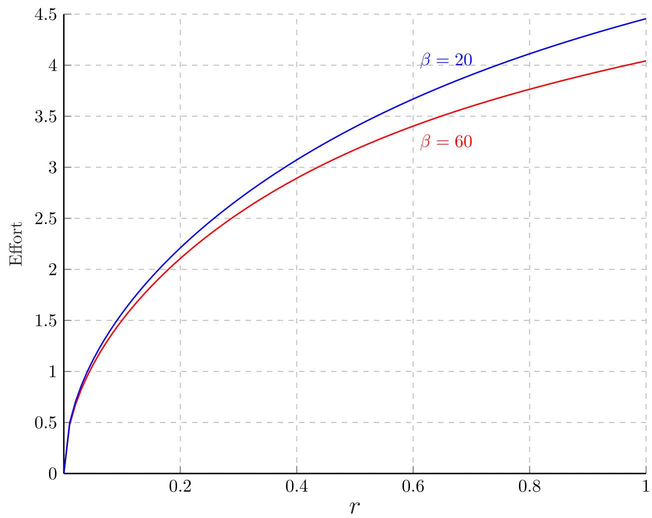


Fig. 5. The relationship between the conflict intensity and r ($\beta > 1$).

3.3. Conflict intensity

Lastly, we consider the conflict intensity, i.e., $na + n\big(b + (n-1)c\big) = na + nb + n(n-1)c$. When $\beta = 1$, we have $\theta = 1$. So the conflict intensity is $(1+n)\big[\frac{nrV}{4}\big]^{\frac{1}{2}}$, which is strictly increasing in r. In the following, we consider the situation when $\beta \neq 1$.

Proposition 6 (*Effect of r on the Conflict Intensity*)**.** *When $\beta > 1$, the conflict intensity $na + nb + n(n-1)c$ is strictly increasing in r. When $\beta < 1$, the conflict intensity may be non-monotonic in r.*

Given $V = 10$ and $n = 2$, Fig. 5 depicts the conflict intensity with β being 20 and 60.

The conflict intensity increases with the level of returns to scale, when the singular agent is weak ($\beta > 1$). The intuition is similar with Proposition 5, in which the competition effect strictly dominates other effects.

When the singular agent is strong compared to the common agents ($\beta < 1$), the relationship between the conflict intensity and the returns to scale technology is subtle, and the specific trend of change may depend on the number of common agents.

Given $V = 10$ and $n = 100$, Fig. 6(a) depicts the conflict intensity with β being 0.01, 0.0001, and 0.000001. (1) When $\beta = 0.01$, the conflict intensity increases with the level of returns to scale. (2) When $\beta = 0.000001$, there exists an inverted U relationship between conflict intensity and the returns to scale technology. (3) When $\beta = 0.0001$, as the level of returns to scale increases, the conflict intensity initially increases, then decreases, and finally increases, or we have a *wave relationship* between the conflict intensity and the returns to scale technology.[9]

Since the total effort of the stronger agent(s) always accounts for a large percentage of the conflict intensity, the conflict intensity tends to have a similar shape with the individual total effort of the stronger agent(s).

The conflict intensity can be used to understand the behaviors of agents in regular networks. Our finding suggests that the conflict intensity may not be increasing with the returns to scale technology in regular networks when agents differ sufficiently in cost asymmetry. In a regular network with cost asymmetry between one individual and others, if the singular agent is stronger, the intensified competition will induce lower conflict intensity. This finding offers a possible explanation for some phenomena observed in real-world conflict networks. In various sports games (e.g., National Basketball Association, Premier League, FIVB Volleyball World Cup, Swiss-system chess tournament, etc.), an extremely strong team or individual will lead to a competitive environment with high pressure for other teams or individuals, resulting in weak competitions. A similar situation can occur in the competition among manufacturers, regions and countries.

4. Conclusion and remarks

This paper investigates the factors that affect agents' behaviors in conflict networks with the returns to scale technology. Focusing on a regular conflict network model, we find that the returns to scale technology plays an important role in affecting equilibrium efforts. Differing from Jiao et al. (2019) who show that network structure asymmetry plays a major role, in a regular network structure, we identify the cost asymmetry as another important factor that determines the relationship between equilibrium efforts of agents and the returns to scale technology.

We show that the individual total effort of each common agent always increases, while the total effort of the singular agent has an inverted U relationship with the level of returns to scale, provided that the singular agent is sufficiently different from common agents in cost functions. Such a result offers a possible explanation for why equilibrium efforts can be reduced as the conflict technology varies. We also show that when the singular agent is not sufficiently different from others in cost functions, the higher level of returns to scale will increase equilibrium efforts. Furthermore, we find that the conflict intensity has a similar shape to the individual total effort of the stronger agent(s).

Our paper mainly focuses on the regular conflict network with a singular agent, while it would be interesting to study whether the inverted U relationship between equilibrium efforts and the returns to scale technology exists under other network structures. In the following, we provide two example of conflicts with a ring structure and a line structure.

Fig. 7 explicitly depicts the ring structure, where individual 1 is the singular agent.

Given $V = 10$, Fig. 8 depicts individual total efforts and the conflict intensity with β being 0.01 and 60. When the singular agent is strong (e.g., $\beta = 0.01$), there exists an inverted U relationship between her total effort X_1 and returns to scale technology r. Meanwhile, the conflict intensity X also has an inverted U relationship with r. On the other hand, when the singular agent is weak (e.g., $\beta = 60$), there still exists an inverted U relationship between her total effort X_1 and the returns to scale technology, but the conflict intensity X increases with r.

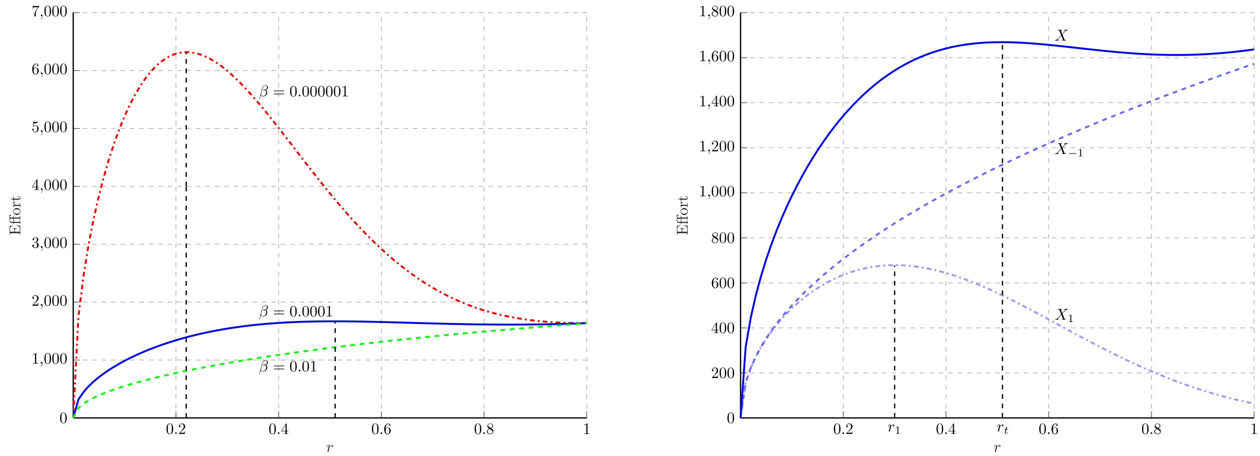
Fig. 9 explicitly depicts a line structure, where agents 2 and 3 have the cost function $\frac{\beta}{2}X_i^2$, and agents 1 and 4 have the cost function $\frac{1}{2}X_j^2$.

Given $V = 10$, Fig. 10 depicts individual total efforts and the conflict intensity with β being 0.005 and 10. Given the competition between agents 2 and 3, individual total efforts of them ($X_2 = X_3$) are always increasing with the returns to scale technology. When the cost asymmetry is sufficiently large, there exists an inverted U relationship between the total efforts of agents 1 and 4 ($X_1 = X_4$) and r.

This example indicates that the inverted U relationship may also occur when the structure asymmetry and the cost asymmetry coexist.

This paper focuses on the decreasing returns to scale technology ($r \leq 1$), which ensures the existence and uniqueness

[9] See Appendix A.7 for more discussion.



(a) Total effort of all agents (b) $\beta = 0.0001$

Fig. 6. The relationship between the conflict intensity and r ($\beta < 1$ and $n = 100$).

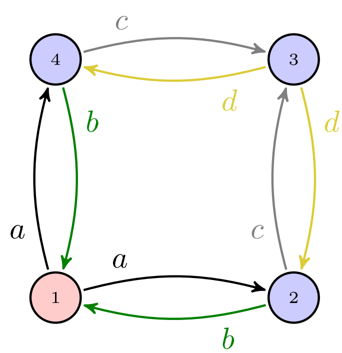


Fig. 7. Ring structure.

of a pure-strategy equilibrium in the regular network model. It is natural to wonder whether and how our main results extend to less noisy contests (namely $r > 1$). We leave it as an open question for future research.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Appendix

A.1. Proof of Proposition 1

Proof of Proposition 1. Based on Xu et al. (2022, Theorem 2 and Lemma 2), there is a unique pure-strategy equilibrium in the game, which is interior. Since each agent i's payoff function u_i is concave in $\boldsymbol{x}_i$, the unique pure-strategy equilibrium $\boldsymbol{x}^* = (\boldsymbol{x}_i^*)_{i\in\mathcal{N}}$ is characterized by the following first-order conditions:

$$V\frac{r(x_{ij}^*)^{r-1}(x_{ji}^*)^r}{[(x_{ij}^*)^r + (x_{ji}^*)^r]^2} - C_i'(X_i^*) = 0,$$

for all $i \in \mathcal{N}$ and all $j \neq i$. Specifically, we have

$$V\frac{r(x_{1j}^*)^{r-1}(x_{j1}^*)^r}{[(x_{1j}^*)^r + (x_{j1})^r]^2} - \beta X_1^* = 0,\ j \neq 1,$$

$$V\frac{r(x_{j1}^*)^{r-1}(x_{1j}^*)^r}{[(x_{j1}^*)^r + (x_{1j}^*)^r]^2} - X_j^* = 0,\ j \neq 1,$$

$$V\frac{r(x_{ij}^*)^{r-1}(x_{ji}^*)^r}{[(x_{ij}^*)^r + (x_{ji}^*)^r]^2} - X_i^* = 0,\ i, j \neq 1,\ i \neq j.$$

In the following, we shall determine a symmetric solution for these equations, which is in turn the unique equilibrium. We let $x_{1j}^* = a$ and $x_{j1}^* = b$ for $j \neq 1$, and $x_{ij}^* = c$ for $i, j \neq 1$ and $i \neq j$. Note that a, b, and c are all positive. Then $X_1^* = na$ and $X_j^* = b+(n-1)c$ for $j \neq 1$. Hence, the first-order conditions becomes

$$V\frac{ra^{r-1}b^r}{(a^r + b^r)^2} - \beta(na) = 0, \tag{2}$$

$$V\frac{rb^{r-1}a^r}{(a^r + b^r)^2} - \big(b + (n-1)c\big) = 0, \tag{3}$$

$$V\frac{r}{4c} - \big(b + (n-1)c\big) = 0. \tag{4}$$

From the Eqs. (2) and (3), we know that $\frac{b}{a} = \frac{n\beta a}{b+(n-1)c}$. Let $\theta = \frac{b}{a} > 0$. Then we have $\frac{n\beta a}{b+(n-1)c} = \theta$, which implies that $c = \frac{n\beta - \theta^2}{(n-1)\theta}a$. From Eqs. (3) and (4), we get that $\frac{\theta^{r-1}}{(1+\theta^r)^2 a} = \frac{1}{4c}$. Since we already have $c = \frac{n\beta-\theta^2}{(n-1)\theta}a$, θ must solve the following equations:

$$\frac{n\beta - \theta^2}{(n-1)\theta} = \frac{(1+\theta^r)^2}{4\theta^{r-1}},$$

or

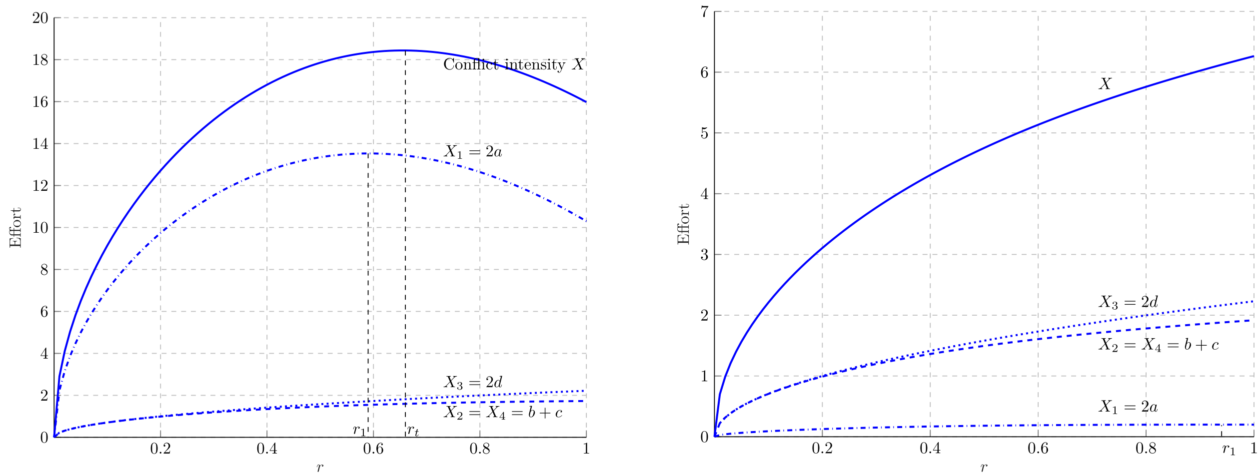
$$\frac{4(n\beta - \theta^2)}{\theta^2} = \frac{(n-1)(1+\theta^r)^2}{\theta^r}, \tag{5}$$

or

$$4(n\beta - \theta^2)\theta^{r-2} - (n-1)(1+\theta^r)^2 = 0.$$

For each $\beta > 0$ and each $r \in (0, 1]$, we will show that the solution of Eq. (5) is unique. Let $\Phi(t, \beta, r) = 4(n\beta - t^2)t^{r-2} - (n-1)(1+t^r)^2$. For any fixed β and r, we have

$$\Phi_t = \frac{\partial\Phi}{\partial t} = -2t^{r-1}\Big(4 + 2(2-r)\frac{n\beta - t^2}{t^2} + (n-1)r(1+t^r)\Big).$$



(a) $\beta = 0.01$ (b) $\beta = 60$

Fig. 8. The relationship between equilibrium efforts and r in the ring structure.

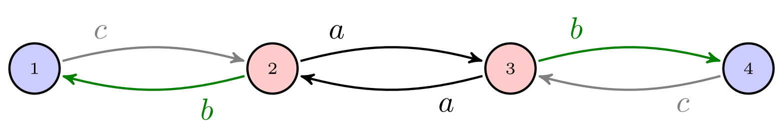
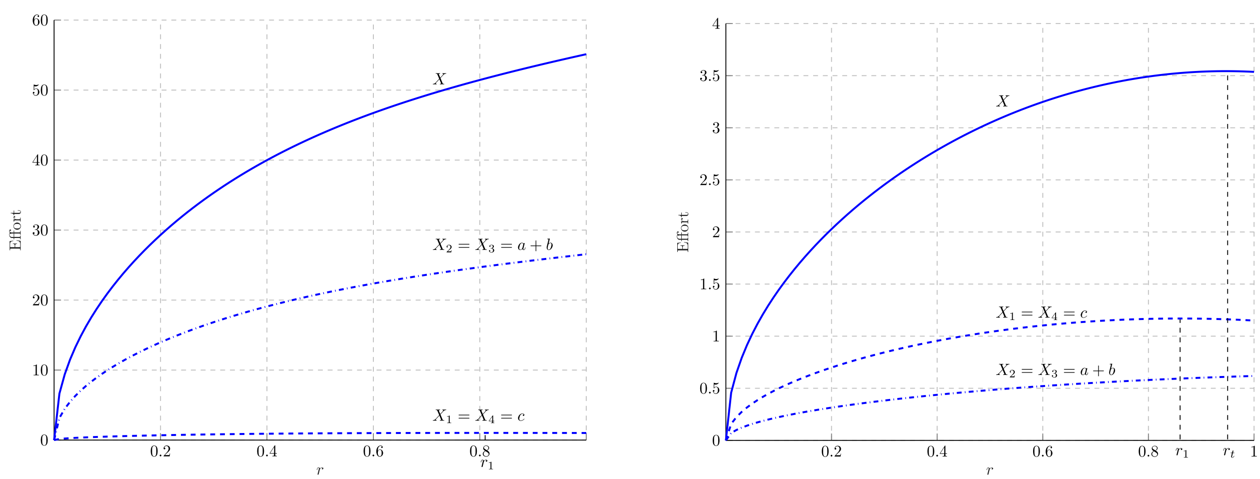


Fig. 9. Line structure.



(a) $\beta = 0.005$ (b) $\beta = 10$

Fig. 10. The relationship between equilibrium efforts and r in the line structure.

Clearly, $\Phi_t < 0$ when $0 < t < \sqrt{n\beta}$. That is, Φ is strictly decreasing in t on $(0, \sqrt{n\beta})$. Since $\lim_{t\downarrow 0} \Phi(t, \beta, r) = +\infty$ and $\Phi(\sqrt{n\beta}, \beta, r) < 0$, the equation $\Phi(t, \beta, r) = 0$ has a unique solution in $(0, \sqrt{n\beta})$, which is denoted by θ. Obviously, $\Phi(t, \beta, r) < 0$ for any $t \geq \sqrt{n\beta}$. It implies that $t = \theta$ is the unique solution of $\Phi(t, \beta, r) = 0$ on $(0, +\infty)$.

Since θ is uniquely determined, the equilibrium effort levels can also be solved, i.e.,

$$a = \left[\frac{(n-1)rV\theta^2}{4n\beta(n\beta-\theta^2)}\right]^{\frac{1}{2}}, \quad b = \theta a = \left[\frac{(n-1)rV\theta^4}{4n\beta(n\beta-\theta^2)}\right]^{\frac{1}{2}}, \text{ and}$$

$$c = \frac{n\beta-\theta^2}{(n-1)\theta}a = \left[\frac{rV(n\beta-\theta^2)}{4n(n-1)\beta}\right]^{\frac{1}{2}}. \quad \square \tag{6}$$

A.2. Proofs of Lemmas 1 and 2

Proof of Lemma 1. Since θ is uniquely determined by Eq. (1), we have that $\theta = 1$ if and only if $\beta = 1$, regardless of r. On one hand, if the solution θ is 1, then obviously β should be 1. On the other hand, if $\beta = 1$, it is easy to see that $\theta = 1$ is a solution. Since Eq. (1) always has the unique solution, the unique solution should be 1. It is clear that $\theta = \frac{b}{a}$ captures the ratio of equilibrium efforts in a bilateral conflict between agents $j \neq 1$ and 1. If $\beta = 1$, then all agents are identical, and hence the equilibrium efforts are the same, or $\theta = 1$.

For each β and each r, we have determined the unique solution $t = \theta = \theta(\beta, r)$ of the equation $\Phi(t, \beta, r) = 4(n\beta - t^2)t^{r-2} -$

$(n-1)(1+t^r)^2 = 0$. Then we have

$$\begin{aligned}\left.\frac{\partial \Phi}{\partial t}\right|_{t=\theta} &= -2\theta^{r-1}\Big(4+2(2-r)\frac{n\beta-\theta^2}{\theta^2}+(n-1)r(1+\theta^r)\Big)\\ &= -2\theta^{r-1}\Big(4+2(2-r)\frac{(n-1)(1+\theta^r)^2}{4\theta^r}\\ &\quad +(n-1)r(1+\theta^r)\Big) \qquad \text{by Eq. (5)}\\ &= -2\theta^{r-1}\frac{(n-1)(2+r)\theta^{2r}+4(n+1)\theta^r+(n-1)(2-r)}{2\theta^r}\\ &= -\frac{1}{\theta}\Big((n-1)(2+r)\theta^{2r}+4(n+1)\theta^r\\ &\quad +(n-1)(2-r)\Big) < 0,\end{aligned}$$

and

$$\left.\frac{\partial \Phi}{\partial \beta}\right|_{t=\theta} = 4n\theta^{r-2} > 0.$$

Thus,

$$\frac{d\theta}{d\beta} = -\left.\frac{\partial \Phi}{\partial \beta}\right|_{t=\theta} \Big/ \left.\frac{\partial \Phi}{\partial t}\right|_{t=\theta} > 0.$$

Since $\theta = 1$ if and only if $\beta = 1$, we have that $\theta > 1$ when $\beta > 1$, and $\theta < 1$ when $0 < \beta < 1$.

We can further narrow down the possible range of $\theta = \theta(\beta, r)$ when $\beta \neq 1$. We know that the RHS of Eq. (5) is bounded below by $4(n-1)$ and bounded above by $\frac{(n-1)(1+\theta)^2}{\theta}$ for $0 < r \leq 1$.[10] Then

$$4(n-1) < \frac{4(n\beta-\theta^2)}{\theta^2} \leq \frac{(n-1)(1+\theta)^2}{\theta}.$$

The first inequality implies that $\theta < \beta^{\frac{1}{2}}$, and the second inequality leads to that $\underline{\theta} \leq \theta$, where $\underline{\theta}$ is the unique positive solution to $(n-1)\theta^3 + 2(n+1)\theta^2 + (n-1)\theta - 4n\beta = 0$.[11]

When $0 < \beta < 1$, we have $\theta < 1$. Hence, $\underline{\theta} < 1$ and $4n\beta = (n-1)\underline{\theta}^3 + 2(n+1)\underline{\theta}^2 + (n-1)\underline{\theta} < (n-1)\underline{\theta} + 2(n+1)\underline{\theta} + (n-1)\underline{\theta} = 4n\underline{\theta}$. Thus, $\beta < \underline{\theta}$. When $\beta > 1$, we have $\theta > 1$. Hence, $\underline{\theta} > 1$ and $4n\beta = (n-1)\underline{\theta}^3 + 2(n+1)\underline{\theta}^2 + (n-1)\underline{\theta} < 4n\underline{\theta}^3$. Thus, $\beta^{\frac{1}{3}} < \underline{\theta}$.

In summary, we have

$$\theta \in \begin{cases}(\beta, \beta^{\frac{1}{2}}), & \text{if } 0 < \beta < 1,\\ (\beta^{\frac{1}{3}}, \beta^{\frac{1}{2}}), & \text{if } \beta > 1. \quad \square\end{cases}$$

Proof of Lemma 2. Recall that $t = \theta = \theta(\beta, r)$ is the unique solution for the equation $\Phi(t, \beta, r) = 4(n\beta - t^2)t^{r-2} - (n-1)(1+t^r)^2 = 0$.

In the proof of Lemma 1, we already have

$$\left.\frac{\partial \Phi}{\partial t}\right|_{t=\theta} = -\frac{1}{\theta}\Big((n-1)(2+r)\theta^{2r}+4(n+1)\theta^r+(n-1)(2-r)\Big) < 0.$$

Moreover, we have

$$\begin{aligned}\left.\frac{\partial \Phi}{\partial r}\right|_{t=\theta} &= \Big(\frac{4(n\beta-\theta^2)}{\theta^2} - 2(n-1)(1+\theta^r)\Big)\theta^r\ln(\theta)\\ &= (n-1)(1+\theta^r)(1-\theta^r)\ln(\theta),\end{aligned}$$

where the second equality is due to Eq. (5). Lemma 1 shows that $\theta = 1$ when $\beta = 1$, $\theta > 1$ when $\beta > 1$, and $\theta < 1$ when $0 < \beta < 1$. Thus, $\frac{\partial \Phi}{\partial r}\big|_{t=\theta} < 0$ whenever $\beta \neq 1$.

Therefore, whenever $\beta \neq 1$,

$$\begin{aligned}\frac{d\theta}{dr} &= -\left.\frac{\partial \Phi}{\partial r}\right|_{t=\theta} \Big/ \left.\frac{\partial \Phi}{\partial t}\right|_{t=\theta}\\ &= \frac{\theta(n-1)(1+\theta^r)(1-\theta^r)\ln(\theta)}{(n-1)(2+r)\theta^{2r}+4(n+1)\theta^r+(n-1)(2-r)} < 0. \qquad (7)\end{aligned}$$

For any $\beta \neq 1$, since θ is decreasing in r, we have $\lim_{r\downarrow 0}\theta = \beta^{\frac{1}{2}}$ and $\lim_{r\uparrow 1}\theta = \underline{\theta}$. $\square$

A.3. Proof of Proposition 2

To consider the effect of r on a, we define the following auxiliary function:

$$\psi(t,r) = 1 + \frac{2[(n-1)t^2+2(n+1)t+(n-1)](1-t)\ln(t)}{(1+t)[(n-1)(2+r)t^2+4(n+1)t+(n-1)(2-r)]}.$$

The following lemma establishes some properties of the function $\psi(t, r)$, which are useful for identifying the effect of r on a.

Lemma 4. *For each $r \in (0, 1]$, the equation $\psi(t, r) = 0$ has a unique solution $\underline{t}_a(r)$ in $(0, 1)$ and a unique solution $\bar{t}_a(r)$ in $(1, +\infty)$. Furthermore, $\psi(t, r) > 0$ when $\underline{t}_a(r) < t < \bar{t}_a(r)$, and $\psi(t, r) < 0$ when $t < \underline{t}_a(r)$ or $t > \bar{t}_a(r)$.*

Proof of Lemma 4. Let $\psi_r = \frac{\partial \psi}{\partial r}$ and $\psi_t = \frac{\partial \psi}{\partial t}$.

We know that ψ_t is given in Box I

Rewrite the above expression as follows

$$\psi_t = \alpha_1(1-t) - \alpha_2\ln(t),$$

where

$$\alpha_1 = \frac{2[(n-1)+2(n+1)t+(n-1)t^2]}{t(1+t)[(n-1)(2-r)+4(n+1)t+(n-1)(2+r)t^2]},$$

$$\alpha_2 = \frac{8[(n-1)r(1-t)^3(1+t)+((n-1)+2(n+1)t+(n-1)t^2)^2]}{(1+t)^2[(n-1)(2-r)+4(n+1)t+(n-1)(2+r)t^2]^2}.$$

It is easy to see that $\alpha_1 > 0$. We can also show that $\alpha_2 > 0$, which follows from

$$\begin{aligned}\alpha_2 &> \frac{8[(n-1)r(1-t)^3(1+t)+(n-1)^2(1+t)^4]}{(1+t)^2[(n-1)(2-r)+4(n+1)t+(n-1)(2+r)t^2]^2}\\ &= \frac{8(n-1)[r(1-t)^3+(n-1)(1+t)^3]}{(1+t)[(n-1)(2-r)+4(n+1)t+(n-1)(2+r)t^2]^2}\\ &> \frac{8(n-1)r[(1-t)^3+(1+t)^3]}{(1+t)[(n-1)(2-r)+4(n+1)t+(n-1)(2+r)t^2]^2}\\ &= \frac{16(n-1)r(1+3t^2)}{(1+t)[(n-1)(2-r)+4(n+1)t+(n-1)(2+r)t^2]^2}\\ &> 0.\end{aligned}$$

Therefore, $\psi_t > 0$ if $0 < t < 1$, and $\psi_t < 0$ if $t > 1$.

We also have that

$$\psi_r = \frac{2(n-1)(1-t)^2[(n-1)t^2+2(n+1)t+(n-1)]\ln(t)}{[(n-1)(2-r)+4(n+1)t+(n-1)(2+r)t^2]^2}.$$

It is clear that $\psi_r > 0$ if $t > 1$, and $\psi_r < 0$ if $0 < t < 1$.

Fix $r \in (0, 1]$. Since ψ is strictly increasing in t on $(0, 1)$ and $\lim_{t\downarrow 0}\psi(t, r) < 0 < \lim_{t\uparrow 1}\psi(t, r)$, the equation $\psi(t, r) = 0$ has a unique solution in $(0, 1)$, denoted by $\underline{t}_a(r)$. Moreover, we have $\psi < 0$ if $t < \underline{t}_a(r)$ and $\psi > 0$ if $\underline{t}_a(r) < t < 1$.

On the other hand, since ψ is strictly decreasing in t on $(1, +\infty)$ and $\lim_{t\downarrow 1}\psi(t, r) > 0 > \lim_{t\uparrow\infty}\psi(t, r)$, the equation $\psi(t, r) = 0$ also has a unique solution in $(1, +\infty)$, denoted by $\bar{t}_a(r)$. Moreover, we have $\psi > 0$ if $1 < t < \bar{t}_a(r)$ and $\psi < 0$ if $t > \bar{t}_a(r)$.

[10] Note that we always have $\frac{(1+\theta)^2}{\theta} \geq \frac{(1+\theta^r)^2}{\theta^r}$ regardless of θ.

[11] When $\theta = 0$, the LHS is negative, and when θ is sufficiently large, the LHS is positive. It is clear that the LHS is strictly increasing in θ. Thus, this equation has a unique positive solution.

$$\psi_t = \frac{2(1-t^2)[(n-1)+2(n+1)t+(n-1)t^2][(n-1)(2-r)+4(n+1)t+(n-1)(2+r)t^2]}{t(1+t)^2[(n-1)(2-r)+4(n+1)t+(n-1)(2+r)t^2]^2} - \frac{8[(n-1)r(1-t)^3(1+t)+((n-1)+2(n+1)t+(n-1)t^2)^2]\ln(t)}{(1+t)^2[(n-1)(2-r)+4(n+1)t+(n-1)(2+r)t^2]^2}.$$

Box I.

In sum, for any $r \in (0, 1]$, we have $\underline{t}_a(r) < 1 < \bar{t}_a(r)$ and

$$\psi \begin{cases} < 0, & \text{if } t < \underline{t}_a(r), \\ = 0, & \text{if } t = \underline{t}_a(r), \\ > 0, & \text{if } \underline{t}_a(r) < t < \bar{t}_a(r), \\ = 0, & \text{if } t = \bar{t}_a(r), \\ < 0, & \text{if } \bar{t}_a(r) < t. \end{cases} \quad \square$$

Proof of Proposition 2. If $\beta = 1$, we know that $\theta = 1$. In this case, $a = b = \left[\frac{rV}{4n}\right]^{\frac{1}{2}}$, both of which are strictly increasing in r. In the following, we focus on the case where $\beta \neq 1$. Let $\underline{\beta}_a = \delta(\underline{t}_a(1)) < 1$ and $\bar{\beta}_a = \delta(\bar{t}_a(1)) > 1$.

Step 1

Recall $a = \left[\frac{(n-1)rV\theta^2}{4n\beta(n\beta-\theta^2)}\right]^{\frac{1}{2}}$. To see the effect of r on a, we just need to look at the effect of r on $g(r,\theta) = \frac{r\theta^2}{n\beta-\theta^2}$.

We have that

$$\begin{aligned}\frac{dg}{dr} &= \frac{\partial g}{\partial r} + \frac{\partial g}{\partial \theta}\frac{d\theta}{dr} \\ &= \frac{\theta^2}{n\beta-\theta^2} + \frac{2n\beta\theta r}{(n\beta-\theta^2)^2}\frac{d\theta}{dr} \\ &= \frac{\theta^2}{n\beta-\theta^2} + \frac{2n\beta\theta r}{(n\beta-\theta^2)^2} \\ &\quad\times \frac{\theta(n-1)(1+\theta^r)(1-\theta^r)\ln(\theta)}{(n-1)(2+r)\theta^{2r}+4(n+1)\theta^r+(n-1)(2-r)} \\ &= \frac{\theta^2}{n\beta-\theta^2}\left[1+\frac{2n\beta r}{n\beta-\theta^2}\right. \\ &\quad\left.\times\frac{(n-1)(1+\theta^r)(1-\theta^r)\ln(\theta)}{(n-1)(2+r)\theta^{2r}+4(n+1)\theta^r+(n-1)(2-r)}\right] \\ &= \frac{\theta^2}{n\beta-\theta^2}\left[1\right. \\ &\quad\left.+\frac{2[(n-1)+2(n+1)\theta^r+(n-1)\theta^{2r}](1-\theta^r)\ln(\theta^r)}{(1+\theta^r)[(n-1)(2+r)\theta^{2r}+4(n+1)\theta^r+(n-1)(2-r)]}\right] \\ &= \frac{\theta^2}{n\beta-\theta^2}\psi(\theta^r, r),\end{aligned}$$

where the third equality follows from Eq. (7) and the last equality is due to the following equation $\frac{n\beta}{n\beta-\theta^2} = \frac{\theta^2}{n\beta-\theta^2}+1 = \frac{4\theta^r}{(n-1)(1+\theta^r)^2}+1 = \frac{(n-1)+2(n+1)\theta^r+(n-1)\theta^{2r}}{(n-1)(1+\theta^r)^2}$, which again follows from Eq. (5).

Since Lemma 1 shows $\theta < \sqrt{\beta}$, we know that $\frac{dg}{dr} > 0$ if and only if $\psi(\theta^r, r) > 0$. By Lemma 4, we know that $\frac{dg}{dr} > 0$ if and only if $\underline{t}_a(r) < \theta^r < \bar{t}_a(r)$.

Step 2

For each $r \in (0, 1]$, we already have $\psi(\underline{t}_a(r), r) = 0$, $\psi(\bar{t}_a(r), r) = 0$, and $\underline{t}_a(r) < 1 < \bar{t}_a(r)$. Hence, Implicit Function Theorem and the proof of Lemma 4 imply that

$$\frac{d\underline{t}_a(r)}{dr} = -\frac{\psi_r|_{t=\underline{t}_a(r)}}{\psi_t|_{t=\underline{t}_a(r)}} > 0 \text{ and } \frac{d\bar{t}_a(r)}{dr} = -\frac{\psi_r|_{t=\bar{t}_a(r)}}{\psi_t|_{t=\bar{t}_a(r)}} > 0.$$

Therefore, both $\underline{t}_a(r)$ and $\bar{t}_a(r)$ are strictly increasing in r. Define

$$\underline{t}_{a0} = \lim_{r\downarrow 0}\underline{t}_a(r),\ \ \underline{t}_{a1} = \lim_{r\uparrow 1}\underline{t}_a(r),\ \ \bar{t}_{a0} = \lim_{r\downarrow 0}\bar{t}_a(r),\ \ \bar{t}_{a1} = \lim_{r\uparrow 1}\bar{t}_a(r).$$

Note that $\lim_{r\downarrow 0}\psi(t,r) = 1 + \frac{(1-t)\ln(t)}{1+t}$, which does not depend on n. Thus, $\underline{t}_{a0}$ and $\bar{t}_{a0}$ should not depend on n either. However, $\underline{t}_{a1}$ and $\bar{t}_{a1}$ both depend on n. Moreover, by solving the equation $0 = \lim_{r\downarrow 0}\psi(t,r) = 1 + \frac{(1-t)\ln(t)}{1+t}$, we have

$$\underline{t}_{a0} = \lim_{r\downarrow 0}\underline{t}_a(r) \approx 0.214 \text{ and } \bar{t}_{a0} = \lim_{r\downarrow 0}\bar{t}_a(r) \approx 4.68.$$

Furthermore, we can narrow down the range of threshold $\bar{t}_{a1}$. Since $\psi(6.5, 1) \approx \frac{-5.05n-140.29}{7.5(153.75n-101.75)} < 0$, we have $\bar{t}_{a1} < 6.5$.

To see the effect of r on θ^r, we have

$$\begin{aligned}\frac{d(\theta^r)}{dr} &= \frac{\partial(\theta^r)}{\partial r} + \frac{\partial(\theta^r)}{\partial\theta}\frac{d\theta}{dr} \\ &= \theta^r\ln(\theta) + r\theta^{r-1}\frac{d\theta}{dr} \\ &= \theta^r\ln(\theta) + r\theta^{r-1} \\ &\quad\times\frac{(n-1)\theta(1+\theta^r)(1-\theta^r)\ln(\theta)}{(n-1)(2+r)\theta^{2r}+4(n+1)\theta^r+(n-1)(2-r)} \\ &= \theta^r\ln(\theta)\frac{2(n-1)\theta^{2r}+4(n+1)\theta^r+2(n-1)}{(n-1)(2+r)\theta^{2r}+4(n+1)\theta^r+(n-1)(2-r)}.\end{aligned}$$

Step 3

Consider the case when $0 < \beta < 1$. Then we have $0 < \theta < 1$, which implies that $\frac{d(\theta^r)}{dr} < 0$. Therefore, as r increases from 0 to 1, θ^r decreases from 1 to $\underline{\theta}$.[12] Note that $\underline{\beta}_a = \delta(\underline{t}_a(1)) = \frac{(n-1)\underline{t}_{a1}^3+2(n+1)\underline{t}_{a1}^2+(n-1)\underline{t}_{a1}}{4n} < 1$.

Case (i): Consider the case when $(1 >)\underline{\theta} \geq \underline{t}_{a1}$. Since $(n-1)\underline{\theta}^3 + 2(n+1)\underline{\theta}^2 + (n-1)\underline{\theta} = 4n\beta$, $\underline{\theta} \geq \underline{t}_{a1}$ is equivalent to $\beta \geq \underline{\beta}_a$.

For any $r \in (0, 1)$, we always have that

$$1 > \theta^r > \underline{\theta} \geq \underline{t}_{a1} > \underline{t}_a(r).$$

That is, for any $r \in (0, 1)$, we have that $\theta^r \in (\underline{t}_a(r), 1) \subseteq (\underline{t}_a(r), \bar{t}_a(r))$, which implies that $\frac{dg}{dr} > 0$.

Case (ii): Consider the case when $\underline{\theta} < \underline{t}_{a1}(< 1)$. Since $(n-1)\underline{\theta}^3 + 2(n+1)\underline{\theta}^2 + (n-1)\underline{\theta} = 4n\beta$, $\underline{\theta} < \underline{t}_{a1}$ is equivalent to $\beta < \underline{\beta}_a$.

We know that

$$\lim_{r\downarrow 0}\theta^r = 1 > \underline{t}_{a0} \text{ and } \lim_{r\uparrow 1}\theta^r = \underline{\theta} < \underline{t}_{a1}.$$

Since θ^r is decreasing in r (from 1 to $\underline{\theta}$) and $\underline{t}_a(r)$ is increasing in r (from $\underline{t}_{a0}$ to $\underline{t}_{a1}$), there must exist a unique r in (0, 1), denoted by r_a, such that $\theta^{r_a} = \underline{t}_a(r_a)$, $\theta^r > \underline{t}_a(r)$ if $r < r_a$, and $\theta^r < \underline{t}_a(r)$ if $r_a < r \leq 1$. As a result, θ^r is between $\underline{t}_a(r)$ and $\bar{t}_a(r)$ when $r < r_a$

[12] Note that $\theta = \underline{\theta}$ is the solution of Eq. (5) when $r = 1$.

and θ^r is outside of $(\underline{t}_a(r), \bar{t}_a(r))$ when $r \geq r_a$. Thus, there exists an inverted U relationship between g and r.

Summary: When $\underline{\beta}_a \leq \beta < 1$, g is increasing in r, and when $\beta < \underline{\beta}_a (< 1)$, there exists an inverted U relationship between g and r.

Step 4

Consider the case when $\beta > 1$. Then we have $\theta > 1$, which implies that $\frac{\mathrm{d}(\theta^r)}{\mathrm{dr}} > 0$. Therefore, as r increases from 0 to 1, θ^r increases from 1 to $\underline{\theta}$. We aim to show that there exists an inverted U relationship between g and r if and only if β is above a certain threshold. We consider the following three cases.

Case (i): If $\underline{\theta} \leq \bar{t}_{a0}$, then for any $r \in (0, 1)$,

$1 < \theta^r < \underline{\theta} \leq \bar{t}_{a0} < \bar{t}_a(r)$.

That is, $\theta^r \in (1, \bar{t}_a(r)) \subseteq (\underline{t}_a(r), \bar{t}_a(r))$ for any $r \in (0, 1)$, which implies that g is increasing in r.

Case (ii): If $\bar{t}_{a0} < \underline{\theta} \leq \bar{t}_{a1}$, we will show that $\theta^r < \bar{t}_a(r)$ always holds for any $r \in (0, 1]$. Given that θ^r increases from 1 to $\underline{\theta}$ as r increases from 0 to 1, and $1 < \bar{t}_{a0} < \underline{\theta}$, there must exist a unique r in (0, 1), denoted by r'_a, such that $\theta^r = \bar{t}_{a0}$ if and only if $r = r'_a$.

Subcase (ii-1). When $r < r'_a$, it is clear that $\theta^r < \bar{t}_{a0} \leq \bar{t}_a(r)$.

Subcase (ii-2). Then it remains to show that when $r \in [r'_a, 1]$, θ^r and $\bar{t}_a(r)$ have no intersection. Since $\underline{\theta} \leq \bar{t}_{a1}$, it is sufficient to show that when $r \in [r'_a, 1]$, the slope of θ^r with respect to r is strictly larger than that of $\bar{t}_a(r)$ with respect to r. We know that

$$
\begin{aligned}
&\frac{\mathrm{d}\bar{t}_a(r)}{\mathrm{dr}} = -\frac{\psi_r|_{t=\bar{t}_a(r)}}{\psi_t|_{t=\bar{t}_a(r)}} \\
&= \frac{(n-1)\bar{t}_a(r)\big(1-\bar{t}_a(r)\big)^2\big(1+\bar{t}_a(r)\big)^2\big[(n-1)\bar{t}_a(r)^2+2(n+1)\bar{t}_a(r)+(n-1)\big]\ln\big(\bar{t}_a(r)\big)}{k_1\big(\bar{t}_a(r)\big)} \\
&< \frac{(n-1)\bar{t}_a(r)\big(1-\bar{t}_a(r)\big)^2\big(1+\bar{t}_a(r)\big)^2\big[(n-1)\bar{t}_a(r)^2+2(n+1)\bar{t}_a(r)+(n-1)\big]\ln\big(\bar{t}_a(r)\big)}{k_2\big(\bar{t}_a(r)\big)} \\
&< \frac{(n-1)\bar{t}_a(r)\big(\bar{t}_a(r)^2-1\big)\ln\big(\bar{t}_a(r)\big)}{2\big[(n-1)\bar{t}_a(r)^2+2(n+1)\bar{t}_a(r)+(n-1)\big]} < \frac{(n-1)\big(\bar{t}_a(r)^2-1\big)\ln\big(\bar{t}_a(r)\big)}{2\big((n-1)\bar{t}_a(r)+2(n+1)\big)} \\
&< \frac{\big(\bar{t}_a(r)^2-1\big)\ln\big(\bar{t}_a(r)\big)}{2\big(\bar{t}_a(r)+2\big)},
\end{aligned}
$$

where

$$
\begin{aligned}
k_1(t) = {} & \big(t^2-1\big)\Big[(n-1)t^2+2(n+1)t+(n-1)\Big] \\
& \times \Big[(n-1)(2+r)t^2+4(n+1)t+(n-1)(2-r)\Big] \\
& + 4t\left[\,(n-1)r\big(1-t\big)^3\big(1+t\big) \right. \\
& \left. + \Big[(n-1)t^2+2(n+1)t+(n-1)\Big]^2\,\right]\ln\big(t\big), \\
k_2(t) = {} & 2\big(t^2-1\big)\Big[(n-1)t^2+2(n+1)t+(n-1)\Big]^2 \\
& + 4t\left[\,(n-1)r\big(1-t\big)^3\big(1+t\big) \right. \\
& \left. + \Big[(n-1)t^2+2(n+1)t+(n-1)\Big]^2\,\right]\ln\big(t\big).
\end{aligned}
$$

Note that the first inequality follows from the fact that $k_1\big(\bar{t}_a(r)\big) - k_2\big(\bar{t}_a(r)\big) = (\bar{t}_a(r)^2-1)\big[(n-1)\bar{t}_a(r)^2+2(n+1)\bar{t}_a(r)+(n-1)\big]r(n-1)(\bar{t}_a(r)^2-1) > 0$ and the second inequality is due to the fact that $(n-1)r(1-\bar{t}_a(r))^3(1+\bar{t}_a(r)) + [(n-1)\bar{t}_a(r)^2+2(n+1)\bar{t}_a(r)+(n-1)]^2 > 0$, which follows from $\alpha_2 > 0$, and the last inequality follows from the fact that $\frac{(n-1)(\bar{t}_a(r)^2-1)\ln(\bar{t}_a(r))}{2[(n-1)\bar{t}_a(r)+2(n+1)]} = \frac{(\bar{t}_a(r)^2-1)\ln(\bar{t}_a(r))}{2\left(\bar{t}_a(r)+2+\frac{4}{n-1}\right)}$ is increasing in n for any fixed $\bar{t}_a(r) > 1$.

In Step 2, we have already shown that $\bar{t}_{a1} < 6.5$. Thus, $\bar{t}_a(r) \leq \bar{t}_{a1} < 6.5$ for each $r \in (0, 1]$. It is easy to derive that $\frac{(\bar{t}_a(r)^2-1)\ln(\bar{t}_a(r))}{2(\bar{t}_a(r)+2)}$ is increasing in $\bar{t}_a(r)$. Thus, we get that $\frac{\mathrm{d}\bar{t}_a(r)}{\mathrm{dr}} < \frac{(6.5^2-1)\ln(6.5)}{2(6.5+2)} < 4.55$. Therefore, the slope of $\bar{t}_a(r)$ with respect to r is strictly smaller than 4.55.

On the other hand,

$$
\begin{aligned}
\frac{\mathrm{d}\left(\theta^r\right)}{\mathrm{dr}} &= \theta^r \frac{\ln(\theta^r)}{r} \\
&\quad \times \frac{2(n-1)\theta^{2r}+4(n+1)\theta^r+2(n-1)}{(n-1)(2+r)\theta^{2r}+4(n+1)\theta^r+(n-1)(2-r)} \\
&> \theta^r \frac{\ln(\theta^r)}{r}\frac{2}{2+r} > \frac{2}{3}\theta^r\ln(\theta^r).
\end{aligned}
$$

Since $\theta^r \geq \underline{\theta}$ for each $r \in [r'_a, 1]$, which is larger than $\bar{t}_{a0} \approx 4.68$, we know that $\frac{\mathrm{d}(\theta^r)}{\mathrm{dr}} > \frac{2}{3} \times 4.68\ln(4.68) > 4.81$.

As a result, we must have $\frac{\mathrm{d}(\theta^r)}{\mathrm{dr}} > \frac{\mathrm{d}\bar{t}_a(r)}{\mathrm{dr}}$ for $r \in [r'_a, 1]$, which implies that there is no intersection of θ^r and $\bar{t}_a(r)$.

Therefore, we always have $(\underline{t}_a(r) < 1 <)\theta^r < \bar{t}_a(r)$ for any $r \in (0, 1]$, which implies that g is increasing in r.

Case (iii): If $\underline{\theta} > \bar{t}_{a1}$, following the same logic as in the previous case, we always have $\theta^r < \bar{t}_a(r)$ when $r < r'_a$. For $r \geq r'_a$, since $\frac{\mathrm{d}(\theta^r)}{\mathrm{dr}} > \frac{\mathrm{d}\bar{t}_a}{\mathrm{dr}}$ and $\underline{\theta} > \bar{t}_{a1}$, there must exist a unique intersection of θ^r and $\bar{t}_a(r)$, which further implies that there exists an inverted U relationship between g and r.

Summary: When $\beta > 1$, the inverted U relationship between g and r exists if and only if $\underline{\theta} > \bar{t}_{a1}$. Recall $\bar{\beta}_a = \frac{(n-1)\bar{t}_{a1}^3+2(n+1)\bar{t}_{a1}^2+(n-1)\bar{t}_{a1}}{4n} > 1$. Since $\underline{\theta}$ is the unique positive solution to $(n-1)\theta^3+2(n+1)\theta^2+(n-1)\theta = 4n\beta$, we have that $\underline{\theta} > \bar{t}_{a1}$ is equivalent to $\beta > \bar{\beta}_a$. Thus, the inverted U relationship between g and r exists if and only if $\beta > \bar{\beta}_a$. □

A.4. Proof of Proposition 3

To consider the effect of r on b, we define the following auxiliary function:

$$
\phi(t,r) = 1 + \frac{4[(n-1)+2nt+(n-1)t^2](1-t)\ln(t)}{(1+t)[(n-1)(2+r)t^2+4(n+1)t+(n-1)(2-r)]}.
$$

The following lemma establishes some properties of the function $\phi(t, r)$, which are useful for identifying the effect of r on b.

Lemma 5. *For each $r \in (0, 1]$, the equation $\phi(t, r) = 0$ has a unique solution $\underline{t}_b(r)$ in (0, 1) and a unique solution $\bar{t}_b(r)$ in $(1, +\infty)$. Furthermore, $\phi(t, r) > 0$ when $\underline{t}_b(r) < t < \bar{t}_b(r)$, and $\phi(t, r) < 0$ when $t < \underline{t}_b(r)$ or $t > \bar{t}_b(r)$.*

Proof of Lemma 5. Recall

$$
\phi(t,r) = 1 + \frac{4[(n-1)+2nt+(n-1)t^2](1-t)\ln(t)}{(1+t)[(n-1)(2+r)t^2+4(n+1)t+(n-1)(2-r)]}.
$$

Let $\phi_r = \frac{\partial\phi}{\partial r}$ and $\phi_t = \frac{\partial\phi}{\partial t}$.

We know that ϕ_t is given in Box II
Rewrite the above expression as follows

$$
\phi_t = \alpha_3(1-t) - \alpha_4\ln(t),
$$

where

$$
\begin{aligned}
\alpha_3 &= \frac{4[(n-1)t^2+2nt+(n-1)]}{t(1+t)[(n-1)(2-r)+4(n+1)t+(n-1)(2+r)t^2]} \\
\alpha_4 &= \frac{8[(n-1)r(1-t)^3(1+t)+2n^2(1+t)^4-2n(1+t)^2(1+t^2)-4t(1-t)^2]}{(1+t)^2[(n-1)(2-r)+4(n+1)t+(n-1)(2+r)t^2]^2}.
\end{aligned}
$$

$$\phi_t = \frac{4(1-t)[(n-1)t^2+2nt+(n-1)]}{t(1+t)[(n-1)(2-r)+4(n+1)t+(n-1)(2+r)t^2]} - \frac{8(n-1)r(1-t)^3(1+t)\ln(t)}{(1+t)^2[(n-1)(2-r)+4(n+1)t+(n-1)(2+r)t^2]^2} - \frac{16[n(n-1)t^4+2(2n+1)(n-1)t^3+2(3n^2-n+2)t^2+2(2n+1)(n-1)t+n(n-1)]\ln(t)}{(1+t)^2[(n-1)(2-r)+4(n+1)t+(n-1)(2+r)t^2]^2}.$$

Box II.

It is clear that $\alpha_3 > 0$. We can also show that $\alpha_4 > 0$, since

$$\alpha_4 = \frac{8[(n-1)r(1-t)^3(1+t)+2n(n-1)(1+t)^4+4nt(1+t)^2-4t(1-t)^2]}{(1+t)^2[(n-1)(2-r)+4(n+1)t+(n-1)(2+r)t^2]^2} > \frac{8(n-1)(1+t)[r(1-t)^3+2n(1+t)^3]}{(1+t)^2[(n-1)(2-r)+4(n+1)t+(n-1)(2+r)t^2]^2} > 0.$$

Therefore, $\phi_t > 0$ if $0 < t < 1$ and $\phi_t < 0$ if $t > 1$.

We also have that

$$\phi_r = \frac{4(n-1)(1-t)^2[(n-1)t^2+2nt+(n-1)]\ln(t)}{[(n-1)(2+r)t^2+4(n+1)t+(n-1)(2-r)]^2}.$$

Clearly, $\phi_r > 0$ if $t > 1$ and $\phi_r < 0$ if $0 < t < 1$.

Fix $r \in (0,1]$. Since ϕ is strictly increasing in t on $(0,1)$ and $\lim_{t_n\downarrow 0}\phi(t,r) < 0 < \lim_{t_n\uparrow 1}\phi(t,r)$, the equation $\phi(t,r)=0$ has a unique solution in $(0,1)$, denoted by $\underline{t}_b(r)$. Moreover, we have $\phi < 0$ if $t < \underline{t}_b(r)$ and $\phi > 0$ if $\underline{t}_b(r) < t < 1$.

On the other hand, since ϕ is strictly decreasing in t on $(1,+\infty)$ and $\lim_{t_n\downarrow 1}\phi(t,r) > 0 > \lim_{t_n\uparrow\infty}\phi(t,r)$, the equation $\phi(t,r)=0$ has a unique solution in $(1,+\infty)$, denoted by $\bar{t}_b(r)$. Moreover, we have $\phi > 0$ if $1 < t < \bar{t}_b(r)$ and $\phi < 0$ if $t > \bar{t}_b(r)$.

In sum, for any $r \in (0,1]$, we have $\underline{t}_b(r) < 1 < \bar{t}_b(r)$ and

$$\phi \begin{cases} <0, & \text{if } t < \underline{t}_b(r), \\ =0, & \text{if } t = \underline{t}_b(r), \\ >0, & \text{if } \underline{t}_b(r) < t < \bar{t}_b(r), \\ =0, & \text{if } t = \bar{t}_b(r), \\ <0, & \text{if } \bar{t}_b(r) < t. \quad \square \end{cases}$$

Proof of Proposition 3. If $\beta = 1$, we know that $\theta = 1$. In this case, $b = \left[\frac{rV}{4n}\right]^{\frac{1}{2}}$, which is strictly increasing in r. In the following, we focus on the case where $\beta \neq 1$. Let $\underline{\beta}_b = \delta(\underline{t}_b(1)) < 1$ and $\bar{\beta}_b = \delta(\bar{t}_b(1)) > 1$

Step 1

Recall $b = \theta\left[\frac{(n-1)rV\theta^2}{4n\beta(n\beta-\theta^2)}\right]^{\frac{1}{2}}$. To see the effect of r on b, we just need to consider the effect of r on $h(r,\theta) = \frac{r\theta^4}{n\beta-\theta^2}$. We have that

$$\begin{aligned}\frac{\mathrm{d}h}{\mathrm{d}r} &= \frac{\partial h}{\partial r} + \frac{\partial h}{\partial\theta}\frac{\mathrm{d}\theta}{\mathrm{d}r} = \frac{\theta^4}{n\beta-\theta^2} + \frac{2r\theta^3(2n\beta-\theta^2)}{(n\beta-\theta^2)^2}\frac{\mathrm{d}\theta}{\mathrm{d}r} \\ &= \frac{\theta^4}{n\beta-\theta^2} + \frac{2r\theta^3(2n\beta-\theta^2)}{(n\beta-\theta^2)^2} \times \frac{\theta(n-1)(1+\theta^r)(1-\theta^r)\ln(\theta)}{(n-1)(2+r)\theta^{2r}+4(n+1)\theta^r+(n-1)(2-r)} \\ &= \frac{\theta^4}{n\beta-\theta^2}\left[1 + \frac{2r(2n\beta-\theta^2)}{n\beta-\theta^2} \times \frac{(n-1)(1+\theta^r)(1-\theta^r)\ln(\theta)}{(n-1)(2+r)\theta^{2r}+4(n+1)\theta^r+(n-1)(2-r)}\right] \\ &= \frac{\theta^4}{n\beta-\theta^2}\left[1 + \frac{4[(n-1)\theta^{2r}+2n\theta^r+(n-1)](1-\theta^r)\ln(\theta^r)}{(1+\theta^r)[(n-1)(2+r)\theta^{2r}+4(n+1)\theta^r+(n-1)(2-r)]}\right],\end{aligned}$$

where the second equality follows from Eq. (7) and the last equality is due to $\frac{2n\beta-\theta^2}{n\beta-\theta^2} = 2 + \frac{\theta^2}{n\beta+\theta^2} = 2 + \frac{4\theta^r}{(n-1)(1+\theta^r)^2} = 2\frac{(n-1)\theta^{2r}+2n\theta^r+(n-1)}{(n-1)(1+\theta^r)^2}$.

Since Lemma 1 shows $\theta < \sqrt{\beta}$, we know that $\frac{\mathrm{dh}}{\mathrm{dr}} > 0$ if and only if $\phi(\theta^r, r) > 0$. By Lemma 5, we further know that $\frac{\mathrm{dh}}{\mathrm{dr}} > 0$ if and only if $\underline{t}_b(r) < \theta^r < \bar{t}_b(r)$.

Step 2

For each $r \in (0,1]$, we already have $\phi(\underline{t}_b(r), r) = 0$, $\phi(\bar{t}_b(r), r) = 0$, and $\underline{t}_b(r) < 1 < \bar{t}_b(r)$. Hence, Implicit Function Theorem and the proof of Lemma 5 imply that

$$\frac{\mathrm{d}\underline{t}_b(r)}{\mathrm{d}r} = -\frac{\phi_r|_{t=\underline{t}_b(r)}}{\phi_t|_{t=\underline{t}_b(r)}} > 0 \text{ and } \frac{\mathrm{d}\bar{t}_b(r)}{\mathrm{d}r} = -\frac{\phi_r|_{t=\bar{t}_b(r)}}{\phi_t|_{t=\bar{t}_b(r)}} > 0.$$

Therefore, both $\underline{t}_b(r)$ and $\bar{t}_b(r)$ are strictly increasing in r. Define

$$\underline{t}_{b0} = \lim_{r\downarrow 0}\underline{t}_b(r),\ \ \underline{t}_{b1} = \lim_{r\uparrow 1}\underline{t}_b(r),\ \ \bar{t}_{b0} = \lim_{r\downarrow 0}\bar{t}_b(r),\ \ \bar{t}_{b1} = \lim_{r\uparrow 1}\bar{t}_{b1}(r).$$

Note that since both $\lim_{r\downarrow 0}\phi(t,r)$ and $\lim_{r\uparrow 0}\phi(t,r)$ depend on n, it is natural that $\underline{t}_{b0}$, $\underline{t}_{b1}$, $\bar{t}_{b0}$, and $\bar{t}_{b1}$ should all depend on n.

We can narrow down the range of $\bar{t}_{b0}$ and $\bar{t}_{b1}$. Notice that $\phi(3.7, 1) \approx \frac{-30.74n-25.16}{4.7(59.87n-27.27)} < 0$. Since ϕ is strictly decreasing in t on $(1,+\infty)$, we have $\bar{t}_{b1} < 3.7$. Moreover, we find that $\phi(2.5, 0) \approx \frac{18.40n-24}{3.5(24.5n-4.5)} > 0$. Since ϕ is strictly decreasing in t on $(1,+\infty)$, we have $\bar{t}_{b0} > 2.5$.

To see the effect of r on θ^r, recall that

$$\frac{\mathrm{d}(\theta^r)}{\mathrm{d}r} = \theta^r\ln(\theta) \times \frac{2(n-1)\theta^{2r}+4(n+1)\theta^r+2(n-1)}{(n-1)(2+r)\theta^{2r}+4(n+1)\theta^r+(n-1)(2-r)}.$$

Step 3

Consider the case when $0 < \beta < 1$. Then we have $0 < \theta < 1$, which implies that $\frac{\mathrm{d}(\theta^r)}{\mathrm{d}r} < 0$. Therefore, as r increases from 0 to 1, θ^r decreases from 1 to $\underline{\theta}$. Let $\underline{\beta}_b = \delta(\underline{t}_b(1)) = \frac{(n-1)\underline{t}_{b1}^3+2(n+1)\underline{t}_{b1}^2+(n-1)\underline{t}_{b1}}{4n} < 1$.

Case (i): Consider the case when $(1>)\underline{\theta} \geq \underline{t}_{b1}$. Since $(n-1)\underline{\theta}^3 + 2(n+1)\underline{\theta}^2 + (n-1)\underline{\theta} = 4n\beta$, $\underline{\theta} \geq \underline{t}_{b1}$ is equivalent to $\beta \geq \underline{\beta}_b$.

For any $r \in (0,1)$, we always have that

$$1 > \theta^r > \underline{\theta} \geq \underline{t}_{b1} > \underline{t}_b(r).$$

That is, for any $r \in (0,1)$, we have that $\theta^r \in (\underline{t}_b(r), 1) \subseteq (\underline{t}_b(r), \bar{t}_b(r))$, which implies that $\frac{\mathrm{dh}}{\mathrm{dr}} > 0$.

Case (ii): Consider the case when $\underline{\theta} < \underline{t}_{b1}(<1)$. Since $(n-1)\underline{\theta}^3 + 2(n+1)\underline{\theta}^2 + (n-1)\underline{\theta} = 4n\beta$, $\underline{\theta} < \underline{t}_{b1}$ is equivalent to $\beta < \underline{\beta}_b$.

We know that

$$\lim_{r\downarrow 0}\theta^r = 1 > \underline{t}_{b0} \text{ and } \lim_{r\uparrow 1}\theta^r = \underline{\theta} < \underline{t}_{b1}.$$

Since θ^r is decreasing in r (from 1 to $\underline{\theta}$) and $\underline{t}_b(r)$ is increasing in r (from $\underline{t}_{b0}$ to $\underline{t}_{b1}$), there must exist s unique r in (0, 1), denoted by r_b, such that $\theta^{r_b} = \underline{t}_b(r_b)$, $\theta^r > \underline{t}_b(r)$ if $r < r_b$, and $\theta^r < \underline{t}_b(r)$ if $r_b < r \leq 1$. As a result, θ^r is between $\underline{t}_b(r)$ and $\bar{t}_b(r)$ when $r < r_b$ and θ^r is outside of $\left(\underline{t}_b(r), \bar{t}_b(r)\right)$ when $r \leq r_b$. Thus, there exists an inverted U relationship between h and r.

Summary: When $\underline{\beta}_b \leq \beta < 1$, h is increasing in r, and when $\beta < \underline{\beta}_b$, there exists an inverted U relationship between h and r.

Step 4

Consider the case when $\beta > 1$. Then we have $\theta > 1$, which implies that $\frac{\mathrm{d}(\theta^r)}{\mathrm{d}r} > 0$. Therefore, as r increases from 0 to 1, θ^r increases from 1 to $\underline{\theta}$. We aim to show that there exists an inverted U relationship between h and r if and only if β is above a certain threshold. We consider the following three cases.

Case (i): If $\underline{\theta} \leq \bar{t}_{b0}$, then for any $r \in (0, 1)$,

$$1 < \theta^r < \underline{\theta} \leq \bar{t}_{b0} < \bar{t}_b(r).$$

That is, $\theta^r \in \left(1, \bar{t}_b(r)\right) \subseteq \left(\underline{t}_b(r), \bar{t}_b(r)\right)$ for any $r \in (0, 1)$, which implies that h is increasing in r.

Case (ii): If $\bar{t}_{b0} < \underline{\theta} \leq \bar{t}_{b1}$, we will show that $\theta^r < \bar{t}_b(r)$ always holds for any $r \in (0, 1]$. Given that θ^r increases from 1 to $\underline{\theta}$ as r increases from 0 to 1, and $1 < \bar{t}_{b0} < \underline{\theta}$, there must exist the unique r, denoted by r'_b, such that $\theta^r = \bar{t}_{b0}$ if and only if $r = r'_b$.

Subcase (ii-1). When $r < r'_b$, it is clear that $\theta^r < \bar{t}_{b0} \leq \bar{t}_b(r)$.

Subcase (ii-2). Then it remains to show that when $r \in [r'_b, 1]$, θ^r and $\bar{t}_b(r)$ have no intersection. Since $\underline{\theta} \leq \bar{t}_{b1}$, it is sufficient to show that when $r \in [r'_b, 1]$, the slope of θ^r with respect to r is strictly larger than that of $\bar{t}_b(r)$ with respect to r. We know that

$$\begin{aligned}
\frac{\mathrm{d}\bar{t}_b(r)}{\mathrm{d}r} &= -\frac{\phi_r|_{t=\bar{t}_b(r)}}{\phi_t|_{t=\bar{t}_b(r)}} \\
&= \frac{(n-1)\bar{t}_b(r)\left(1-\bar{t}_b(r)\right)^2\left(1+\bar{t}_b(r)\right)^2[(n-1)\bar{t}_b(r)^2+2n\bar{t}_b(r)+(n-1)]\ln\left(\bar{t}_b(r)\right)}{k_3\left(\bar{t}_b(r)\right)} \\
&< \frac{(n-1)\bar{t}_b(r)\left(1-\bar{t}_b(r)\right)^2\left(1+\bar{t}_b(r)\right)^2[(n-1)\bar{t}_b(r)^2+2n\bar{t}_b(r)+(n-1)]\ln\left(\bar{t}_b(r)\right)}{k_4\left(\bar{t}_b(r)\right)} \\
&< \frac{(n-1)\bar{t}_b(r)\left(\bar{t}_b(r)^2-1\right)\ln\left(\bar{t}_b(r)\right)}{2[(n-1)\bar{t}_b(r)^2+2n\bar{t}_b(r)+(n-1)]} < \frac{(n-1)\left(\bar{t}_b(r)^2-1\right)\ln\left(\bar{t}_b(r)\right)}{2[(n-1)\bar{t}_b(r)+2n]} \\
&< \frac{\left(\bar{t}_b(r)^2-1\right)\ln\left(\bar{t}_b(r)\right)}{2(\bar{t}_b(r)+2)}
\end{aligned}$$

where

$$\begin{aligned}
k_3(t) = (t^2-1)&\left[(n-1)+2nt+(n-1)t^2\right] \\
&\times\left[(n-1)(2+r)t^2+4(n+1)t+(n-1)(2-r)\right] \\
&+ 2t\left[(n-1)r(1-t)^3(1+t)+2n^2(1+t)^4\right. \\
&\left.\quad -2n(1+t)^2(1+t^2)-4t(1-t)^2\right]\ln(t) \\
k_4(t) = 2(t^2-1)&\left[(n-1)+2nt+(n-1)t^2\right] \\
&\times\left[(n-1)+2(n+1)t+(n-1)t^2\right] \\
&+ 2t\left[(n-1)r(1-t)^3(1+t)+2n^2(1+t)^4\right. \\
&\left.\quad -2n(1+t)^2(1+t^2)-4t(1-t)^2\right]\ln(t).
\end{aligned}$$

Note that the first inequality follows from $k_3\left(\bar{t}_b(r)\right) > k_4\left(\bar{t}_b(r)\right)$ and the last inequality follows from the fact that $\frac{(n-1)(\bar{t}_b(r)^2-1)\ln(\bar{t}_b(r))}{2[(n-1)\bar{t}_b(r)+2n]} = \frac{(\bar{t}_b(r)^2-1)\ln(\bar{t}_b(r))}{2[\bar{t}_b(r)+2+\frac{2}{n-1}]}$ is increasing in n for any fixed $\bar{t}_b(r) > 1$.

In Step 2, we have already shown that $\bar{t}_{b1} < 3.7$. Thus, $\bar{t}_b(r) \leq \bar{t}_{b1} < 3.7$ for each $r \in (0, 1]$. It is easy to derive that $\frac{(\bar{t}_b(r)^2-1)\ln(\bar{t}_b(r))}{2(\bar{t}_b(r)+2)}$ is increasing in $\bar{t}_b(r)$. Thus, we get that $\frac{\mathrm{d}\bar{t}_b(r)}{\mathrm{d}r} < \frac{(3.7^2-1)\ln(3.7)}{2(3.7+2)} < 1.46$. Therefore, the slope of $\bar{t}_b(r)$ with respect to r is strictly smaller than 1.46.

On the other hand, we already have $\bar{t}_{b0} > 2.5$. Thus, $\frac{\mathrm{d}(\theta^r)}{\mathrm{d}r} > \frac{2}{3}\theta^r\ln(\theta^r) > \frac{2}{3}\bar{t}_{b0}\ln(\bar{t}_{b0}) > 1.52$.

As a result, we must have $\frac{\mathrm{d}(\theta^r)}{\mathrm{d}r} > \frac{\mathrm{d}\bar{t}_b(r)}{\mathrm{d}r}$ for $r \in [r'_b, 1]$, which implies that there is no intersection of θ^r and $\bar{t}_b(r)$.

Therefore, we always have $(\underline{t}_b(r) < 1 <)\theta^r < \bar{t}_b(r)$ for any $r \in (0, 1]$, which implies that h is increasing in r.

Case (iii): If $\underline{\theta} > \bar{t}_{b1}$, following the same logic as in the previous case, we always have $\theta^r < \bar{t}_b(r)$ when $r < r'_b$. For $r \geq r'_b$, since $\frac{\mathrm{d}(\theta^r)}{\mathrm{d}r} > \frac{\mathrm{d}\bar{t}_b(r)}{\mathrm{d}r}$ and $\underline{\theta} > \bar{t}_{b1}$, there must exist a unique intersection of θ^r and $\bar{t}_b(r)$, which further implies that there exists an inverted U relationship between h and r.

Summary: When $\beta > 1$, the inverted U relationship between h and r exists if and only if $\underline{\theta} > \bar{t}_{b1}$. Recall $\bar{\beta}_b = \frac{(n-1)\bar{t}_{b1}^3+2(n+1)\bar{t}_{b1}^2+(n-1)\bar{t}_{b1}}{4n}$. Since $\underline{\theta}$ is the unique positive solution to $(n-1)\theta^3+2(n+1)\theta^2+(n-1)\theta = 4n\beta$, we have that $\underline{\theta} > \bar{t}_{b1}$ is equivalent to $\beta > \bar{\beta}_b$. Thus, the inverted U relationship between h and r exists if and only if $\beta > \bar{\beta}_b$. □

A.5. Proof of Proposition 4

Proof of Proposition 4. Recall that $c = \left[\frac{rV(n\beta-\theta^2)}{4n(n-1)\beta}\right]^{\frac{1}{2}}$. When $\beta = 1$, $c = \left[\frac{rV}{4n}\right]^{\frac{1}{2}}$, which is strictly increasing in r. To see the effect of r on c when $\beta \neq 1$, we just need to look at the effect of r on $f(r,\theta) = r(n\beta-\theta^2)$.

Note that

$$\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}r} &= \frac{\partial f}{\partial r} + \frac{\partial f}{\partial \theta}\frac{\mathrm{d}\theta}{\mathrm{d}r} \\
&= n\beta - \theta^2 - 2r\theta\frac{\mathrm{d}\theta}{\mathrm{d}r} \\
&= n\beta - \theta^2 \\
&\quad - 2r\theta\frac{\theta(n-1)(1+\theta^r)(1-\theta^r)\ln(\theta)}{(n-1)(2+r)\theta^{2r}+4(n+1)\theta^r+(n-1)(2-r)} \\
&= \frac{(n-1)\theta^2(1+\theta^r)^2}{4\theta^r} \\
&\quad - \frac{2(n-1)\theta^2(1+\theta^r)(1-\theta^r)\ln(\theta^r)}{(n-1)(2+r)\theta^{2r}+4(n+1)\theta^r+(n-1)(2-r)} \\
&= (n-1)\theta^2(1+\theta^r)\left[\frac{1+\theta^r}{4\theta^r}\right. \\
&\quad \left.- \frac{2(1-\theta^r)\ln(\theta^r)}{(n-1)(2+r)\theta^{2r}+4(n+1)\theta^r+(n-1)(2-r)}\right],
\end{aligned}$$

where the third equality follows from Eq. (7). It is clear that $\frac{\mathrm{d}f}{\mathrm{d}r} > 0$ for any $r \in (0, 1]$, and c is strictly increasing in r. □

A.6. Proofs of Lemma 3, Propositions 5, and 6

Proof of Lemma 3. It is straightforward to have that

$$\frac{\mathrm{d}b}{\mathrm{d}r} = \frac{\mathrm{d}\theta}{\mathrm{d}r}a + \theta\frac{\mathrm{d}a}{\mathrm{d}r} < \theta\frac{\mathrm{d}a}{\mathrm{d}r}.$$

Thus, when a is decreasing in r, b must also be decreasing in r. □

Proof of Proposition 5. Note that

$$b+(n-1)c = \frac{n\beta}{\theta}a = \frac{n\beta}{\theta}\left[\frac{(n-1)rV\theta^2}{4n\beta(n\beta-\theta^2)}\right]^{\frac{1}{2}} = \left[\frac{(n-1)n\beta rV}{4(n\beta-\theta^2)}\right]^{\frac{1}{2}}.$$

When $\beta = 1$, then $\theta = 1$, and hence $b+(n-1)c = \left[\frac{nrV}{4}\right]^{\frac{1}{2}}$, which is strictly increasing in r. In the following, we focus on the case when $\beta \neq 1$.

To see the effect of r on $b+(n-1)c$, it suffices to consider the effect of r on $y(r,\theta) = \frac{r}{n\beta-\theta^2}$. We have that

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}r} &= \frac{\partial y}{\partial r} + \frac{\partial y}{\partial \theta}\frac{\mathrm{d}\theta}{\mathrm{d}r}\\ &= \frac{1}{n\beta-\theta^2} + \frac{2\theta r}{(n\beta-\theta^2)^2}\frac{\mathrm{d}\theta}{\mathrm{d}r}\\ &= \frac{1}{n\beta-\theta^2} + \frac{2\theta r}{(n\beta-\theta^2)^2}\\ &\quad\times \frac{\theta(n-1)(1+\theta^r)(1-\theta^r)\ln(\theta)}{(n-1)(2+r)\theta^{2r}+4(n+1)\theta^r+(n-1)(2-r)}\\ &= \frac{1}{n\beta-\theta^2}\left[1+\frac{8\theta^r}{1+\theta^r}\right.\\ &\quad\left.\times \frac{(1-\theta^r)\ln(\theta^r)}{(n-1)(2+r)\theta^{2r}+4(n+1)\theta^r+(n-1)(2-r)}\right],\end{aligned}$$

where the third and fourth equalities follow from Eqs. (7) and (5), respectively.

Let

$$\varphi(t,r) = 1 + \frac{8t(1-t)\ln(t)}{(1+t)\big[(n-1)(2+r)t^2+4(n+1)t+(n-1)(2-r)\big]}.$$

It is shown in Lemma 1 that $\theta < \sqrt{\beta}$, so we know that $\frac{\mathrm{d}y}{\mathrm{d}r} > 0$ if and only if $\varphi(\theta^r, r) > 0$. It is easy to see that when $t > 0$ and $r \in (0,1]$,

$$\begin{aligned}\varphi(t,r) &= 1 + \frac{8t(1-t)\ln(t)}{(1+t)\big[(n-1)(2+r)t^2+4(n+1)t+(n-1)(2-r)\big]}\\ &\geq 1+\frac{8t(1-t)\ln(t)}{(1+t)\big[(2+r)t^2+12t+(2-r)\big]},\end{aligned}$$

where the last expression is denoted by $\underline{\varphi}(t,r)$. We shall show that $\underline{\varphi}(t,r) > 0$ for each $t > 0$ and each $r \in (0,1]$. If $t = 1$, then $\underline{\varphi}(1,r) = 1 > 0$ no matter of the value of r.

It is easy to see that

$$\frac{\partial \underline{\varphi}}{\partial r} = \frac{8(1-t)^2 t\ln(t)}{[(2+r)t^2+12t+2-r]^2}.$$

Obviously, $\frac{\partial \underline{\varphi}}{\partial r} > 0$ when $t > 1$, and $\frac{\partial \underline{\varphi}}{\partial r} < 0$ when $0 < t < 1$.

We first consider the case when $t > 1$. In this case, we have $\frac{\partial \underline{\varphi}}{\partial r} > 0$. Thus, $\underline{\varphi}(t,r) \geq \underline{\varphi}(t,0) = 1-\frac{4t(t-1)\ln(t)}{(t+1)[t^2+6t+1]} > 1-\frac{4\ln(t)}{t+6}$. To show that $1-\frac{4\ln(t)}{t+6} > 0$ on $t \in (1,+\infty)$, we just need to show that $t+6-4\ln(t) > 0$ on $t \in (1,+\infty)$. It is easy to see that $t+6-4\ln(t)$ is minimized at $t = 4$ on $t \in (1,+\infty)$ and the minimal is $10-4\ln(4) \approx 4.45$.

We next consider the case when $0 < t < 1$. Then $\frac{\partial \underline{\varphi}}{\partial r} < 0$. Thus, $\underline{\varphi}(t,r) \geq \underline{\varphi}(t,1) = 1+\frac{8t(1-t)\ln(t)}{(1+t)[3t^2+12t+1]} > 1+\frac{8t\ln(t)}{12t+1}$. To show that $1+\frac{8t\ln(t)}{12t+1} > 0$ on $t \in (0,1)$, we just need to show that $\frac{12t+1}{8t}+\ln(t) > 0$ on $t \in (0,1)$. It is easy to show that $\frac{12t+1}{8t}+\ln(t)$ is minimized at $t = \frac{1}{8}$ on $t \in (0,1)$, and the minimal is $\frac{5}{2}-\ln(8) \approx 0.42$.

Therefore, $\frac{\mathrm{d}y}{\mathrm{d}r}$ is always positive, that is, $b+(n-1)c$ is strictly increasing in $r \in (0,1]$. □

Proof of Proposition 6. By Lemma 1, we have that $\beta > 1$ if and only if $\theta > 1$.

To see the effect of r on the total effort of all the agents, it suffices to see the effect of r on $z(r,\theta) = \frac{r(\theta+n\beta)^2}{n\beta-\theta^2}$. We have that

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}r} &= \frac{\partial z}{\partial r}+\frac{\partial z}{\partial \theta}\frac{\mathrm{d}\theta}{\mathrm{d}r}\\ &= \frac{(\theta+n\beta)^2}{n\beta-\theta^2}+\frac{2r(\theta+n\beta)[n\beta(1+\theta)]}{(n\beta-\theta^2)^2}\frac{\mathrm{d}\theta}{\mathrm{d}r}\\ &= \frac{(\theta+n\beta)^2}{n\beta-\theta^2}\left[1+\frac{2r[n\beta(1+\theta)]}{(n\beta-\theta^2)(\theta+n\beta)}\right.\\ &\quad\left.\times\frac{\theta(n-1)(1+\theta^r)(1-\theta^r)\ln(\theta)}{(n-1)(2+r)\theta^{2r}+4(n+1)\theta^r+(n-1)(2-r)}\right]\\ &= \frac{(\theta+n\beta)^2}{n\beta-\theta^2}\frac{\zeta(r,\theta)}{\zeta_0(r,\theta)},\end{aligned}$$

where

$$\begin{aligned}\zeta_0(r,\theta) &= (1+\theta^r)\big[(n-1)(1+\theta^r)^2+4\theta^r+4\theta^{r-1}\big]\\ &\quad\times\big[(n-1)(2+r)\theta^{2r}+4(n+1)\theta^r+(n-1)(2-r)\big],\\ \zeta(r,\theta) &= \zeta_0(r,\theta)+8\big[(n-1)+2(n+1)\theta^r+(n-1)\theta^{2r}\big]\\ &\quad\times(\theta^{r-1}+\theta^r)(1-\theta^r)\ln(\theta^r).\end{aligned}$$

Note that the third and fourth equalities follow from Eqs. (7) and (5), respectively.

It is shown in Lemma 1 that $\theta < \sqrt{\beta}$. Clearly, $\zeta_0(r,\theta) > 0$ when $\theta > 1$. Thus, $\frac{\mathrm{d}z}{\mathrm{d}r} > 0$ if and only if $\zeta(r,\theta) > 0$.

Since $r\theta^{2r}-r = r(\theta^{2r}-1) > 1$, we have that

$$\begin{aligned}&\zeta_0(r,\theta)\\ &= (1+\theta^r)\big[(n-1)(1+\theta^r)^2+4\theta^r+4\theta^{r-1}\big]\\ &\quad\times\big[(n-1)(2+r)\theta^{2r}+4(n+1)\theta^r+(n-1)(2-r)\big]\\ &> (1+\theta^r)\big[(n-1)(1+\theta^r)^2+4\theta^r+4\theta^{r-1}\big]\\ &\quad\times\big[2(n-1)\theta^{2r}+4(n+1)\theta^r+2(n-1)\big].\end{aligned}$$

Therefore,

$$\begin{aligned}\zeta(r,\theta) &= \zeta_0(r,\theta)+8\big[(n-1)+2(n+1)\theta^r+(n-1)\theta^{2r}\big]\\ &\quad\times(\theta^{r-1}+\theta^r)(1-\theta^r)\ln(\theta^r)\\ &> (1+\theta^r)\big[(n-1)(1+\theta^r)^2+4\theta^r+4\theta^{r-1}\big]\\ &\quad\times\big[2(n-1)\theta^{2r}+4(n+1)\theta^r+2(n-1)\big]\\ &\quad+4\big[2(n-1)\theta^{2r}+4(n+1)\theta^r+2(n-1)\big]\\ &\quad\times(\theta^{r-1}+\theta^r)(1-\theta^r)\ln(\theta^r)\\ &= \big[2(n-1)\theta^{2r}+4(n+1)\theta^r+2(n-1)\big]\zeta_1(r,\theta),\end{aligned}$$

where $\zeta_1(r,\theta) = (n-1)(1+\theta^r)^3+4(\theta^{r-1}+\theta^r)\big[1+\theta^r+\ln(\theta^r)-\theta^r\ln(\theta^r)\big]$. Then we have that

$$\begin{aligned}&\zeta_1(r,\theta) > (1+\theta^r)^3+4(\theta^{r-1}+\theta^r)\\ &\quad\times\big[1+\theta^r+\ln(\theta^r)-\theta^r\ln(\theta^r)\big]\\ &= \theta^{3r}+7\theta^{2r}+7\theta^r+1+\underbrace{4\theta^{r-1}+4\theta^{2r-1}}_{>0}+4\theta^r\ln(\theta^r)\\ &\quad+\underbrace{4\theta^{r-1}\ln(\theta^r)}_{>0}-\underbrace{4\theta^{2r-1}\ln(\theta^r)}_{<4\theta^{2r}\ln(\theta^r)}-4\theta^{2r}\ln(\theta^r)\\ &> \theta^{3r}+7\theta^{2r}+7\theta^r+1+4\theta^r\ln(\theta^r)-8\theta^{2r}\ln(\theta^r).\end{aligned}$$

It is easy to find that the function $\underline{\zeta}(x) = x^3+7x^2+7x+1+4x\ln(x)-8x^2\ln(x) > 0$ when $x > 1$. Hence, $\zeta(r,\theta) > 0$ when $\beta > 1$, which means that the total effort is strictly increasing in r. □

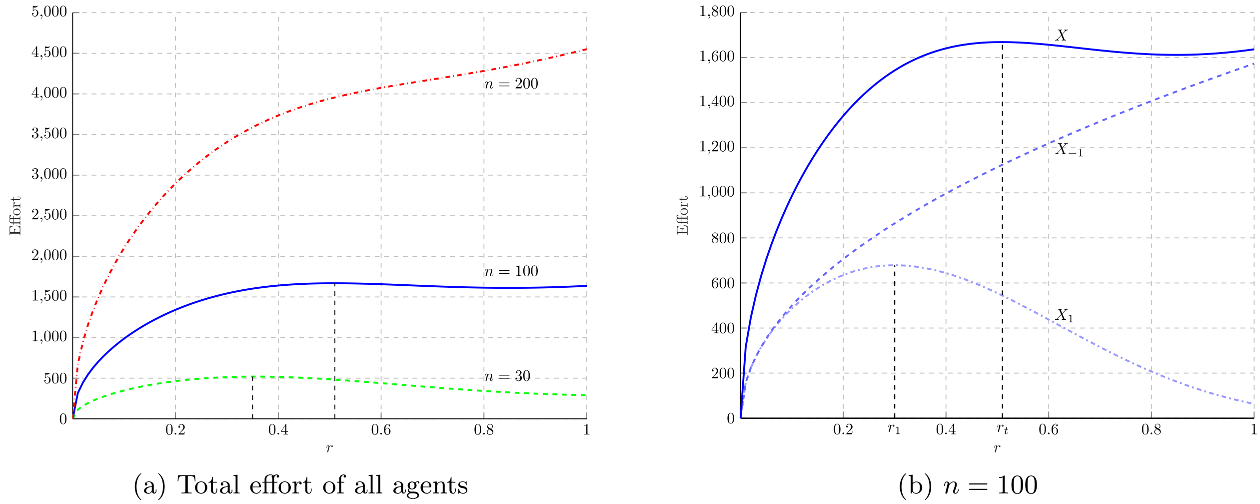


Fig. 11. The relationship between the conflict intensity and r ($\beta = 0.0001$ and $n = 30, 100, 200$).

A.7. Wave relationship

We would like to provide more discussion on the relationship between the returns to scale technology r and the conflict intensity. According to the numerical example shown in Fig. 6, one may conjecture that given $\beta < 1$ and a relatively large integer n, there exist two cutoff points β_1 and β_2 with $0 < \beta_1 < \beta_2 < 1$: (1) When $0 < \beta < \beta_1$, there exists an inverted U relationship between conflict intensity and returns to scale technology; (2) When $\beta \in (\beta_1, \beta_2)$, we have a wave relationship in which the conflict intensity first increases, then decreases, and finally increases with the level of returns to scale; (3) When $1 > \beta > \beta_2$, the conflict intensity increases with the level of returns to scale.

Intuitively, with an extremely small β (e.g., $\beta \in (\beta_1, \beta_2)$), the singular agent has a cost advantage and a population disadvantage. The competition effect dominates for all agents under a low level of returns to scale. While when the level of returns to scale increases to a moderate level, the discouragement effect between the singular agent and common agents dominates, causing the equilibrium efforts of those groups to decrease, and therefore the total effort to decrease. However, when the level of returns to scale further increases to a high level, although the equilibrium effort of (or against) the singular agent (a and b) keeps decreasing, due to the population advantage, the competition effect among the common agents themselves ($(n-1)c$) is strictly dominant the discouragement effect from (or against) the singular agent, hence the total effort eventually increases. Figs. 6(b) gives the details for the equilibrium efforts of all the agents, the center, and all the others except the center when $\beta = 0.0001$ and $n = 100$.

Moreover, when the number of common agents n increases, the second critical point of returns to scale for total effort will appear earlier since the population advantage becomes more apparent. Therefore, when the number of common agents is small, the second critical point of r is greater than 1. As a result, on the interval $r \in (0, 1]$, no wave relationship exists, and we can only observe an inverted U. When the number of common agents increases, the first critical point of returns to scale for total effort will appear earlier, which is also due to the huge population disadvantage. This follows from conventional wisdom.

The following figure illustrates another numerical example.

Given $V = 10$ and $\beta = 0.0001$, Fig. 11(a) depicts the conflict intensity with n being 30, 100, and 200. (1) When $n = 200$, the conflict intensity increases with the level of returns to scale. (2) When $n = 30$, there exists an inverted U relationship between the conflict intensity and returns to scale technology. (3) When $n = 100$, we have a wave relationship between the conflict intensity and returns to scale technology.

One may also conjecture that given a relatively small $\beta < 1$, there exist two cutoff points n_1 and n_2 with $n_1 < n_2$: (1) When $n < n_1$, there exists an inverted U relationship between the conflict intensity and returns to scale technology; (2) When $n \in (n_1, n_2)$, we have a wave relationship; (3) When $n > n_2$, the conflict intensity increases with the level of returns to scale.

From this, it may be inferred that the wave relationship occurs only when β and n are relatively balanced; that is, β is moderately small with respect to n and n is moderately large with response to β.

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