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Bipartite conflict networks with returns to scale technology☆

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ABSTRACT

This paper considers a complete bipartite conflict network model with returns to scale technology. We show that the impact of the returns to scale technology on agents' equilibrium behaviors can be determined by two important factors: the size ratio of two coalitions and the degree of local interdependencies of conflicts. We find an inverted U relationship between the returns to scale technology and (individual and total) equilibrium efforts when two coalitions differ sufficiently in size and the degree of local interdependencies of conflicts is high. Our finding suggests that an increase in the returns to scale technology can be beneficial in the sense that it reduces conflict intensity.

1. Introduction

Conflict refers to certain form of friction or discord when the beliefs or actions of agents are in opposition. It has been widely studied in the contest framework.[1] To the best of our knowledge, most of the classical contest-based literature focuses on stylized models of one or more isolated and independent conflicts. However, the parties to a conflict are often involved in more than one conflict in the same period of time. For instance, conflict in the form of war among different opposing military alliances is probably the most important example in which agents need to simultaneously deal with several conflicts. Leaders of different political parties are frequently involved in multiple political campaigns. Members of different groups, which differ in ideology, religion or even race, are often unfriendly to one another and sometimes behave as

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[1] See the review by Kimbrough et al. (2017).

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enemies. In addition, the market competition between a large company that diversifies its product line and individual sellers that offer different categories of products can also be analyzed using a framework of conflict networks. All the examples above can be considered conflicts that exist within network structures. Unlike traditional models of isolated/independent conflicts, in such conflict networks, local interdependencies between distinct conflicts can have substantial implications for the behaviors of conflict parties and the intensity of conflicts.

In a recent paper, Franke and Öztürk (2015) develop a model of conflict networks that captures locally interdependent conflicts among agents. In their model, each agent is involved in several bilateral conflicts and decides simultaneously how much to invest in each conflict. The marginal cost of investment in one conflict depends on the investments in all other conflicts made by the agent. An agent's probability of winning against a particular rival depends on her investment in certain conflict-specific technology. By focusing on a specific contest model proposed by Tullock (1980), Franke and Öztürk (2015) provide an equilibrium analysis of two classes of conflict network structures (regular networks and complete bipartite networks). They find that the aggregate effort in a conflict network, which they call "conflict intensity", is affected by the structure of networks (e.g., the number of agents, the classes of conflict networks).

In addition to network structures, other factors can influence how parties interact with each other in conflict networks, with the conflict technology being a particularly important one. In a general Tullock contest model, the contest technology is measured by the degree of returns to scale. Franke and Öztürk (2015) consider a special Tullock contest, known as the lottery contest, which exhibits constant returns to scale. However, different conflicts vary in the degree of returns to scale in their technologies. For instance, modern high-tech wars may provide a much higher rate of return than did ancient wars. The return on investments to influence voters tends to be the highest during the key period of national election. In the case of market competition, modern sales modes (such as the internet, e-business and online sales) seem more effective and provide a higher rate of return than traditional offline sales. An interesting question, therefore, is how conflict technology affects the behavior of agents in conflict networks. Our paper makes the first attempt to address this question.

We consider a conflict network model adapted from Franke and Öztürk (2015) by incorporating returns to scale technology. Instead of analyzing a general network structure, we focus on one of the most classical networks: complete bipartite conflict networks. A particular feature of this type of networks is the group size difference between two coalitions. This asymmetric characteristic plays a key role in affecting the relationship between agents' equilibrium behavior and returns to scale technology. We first characterize a unique pure-strategy equilibrium, in which each agent in the same coalition exerts the same (positive) level of effort in each bilateral conflict. We further analyze how the conflict technology affects the equilibrium behavior of each agent. Our main finding is that as the returns to scale technology increases, conflict parties do not necessarily invest more to win conflicts. We identify two main factors that determine the relationship between equilibrium efforts of agents and the returns to scale technology: the size ratio of two coalitions and the degree of local interdependencies of conflicts. When two coalitions differ sufficiently in size and the degree of local interdependencies of conflicts is high, there exists an inverted U relationship between the equilibrium effort of each agent and the level of returns to scale. That is, as each bilateral conflict becomes fiercer, agents tend to invest more under a low level of returns to scale but less under a high level of returns to scale. By contrast, when the two coalitions are close in size or when the degree of local interdependencies of conflicts is low, the equilibrium effort of each agent is positively related to the level of returns to scale.

To explain the results above, we identify two effects of conflict technology on agents' incentives to exert efforts: a competition effect and a discouragement effect. On the one hand, each agent has more incentive to exert effort as each conflict becomes more discriminatory (competition effect). On the other hand, because of the asymmetry in group sizes and the local interdependencies of conflicts, an agent in the smaller coalition always needs to fight with more rivals so that the effective cost in each conflict is higher than that of any rival in the other coalition. When each conflict becomes less noisy, an agent in the smaller coalition tends to exert less effort, since it is less likely that she will win when competing with more rivals (discouragement effect). While these two effects work in opposite directions, the net effect would then depend on which one dominates. When the asymmetry between the two coalitions is large, which occurs when they differ sufficiently in size and the degree of local interdependencies of conflicts is high, the competition effect dominates under a low level of returns to scale, while the discouragement effect dominates under a high level of returns to scale. This results in an inverted U relationship between the equilibrium effort of each agent and the level of returns to scale. Otherwise, the competition effect always dominates, implying a positive relationship between the equilibrium effort of each agent and the level of returns to scale.

Provided that conflict intensity is the aggregation of individual efforts, we further establish an inverted U relationship between returns to scale technology and conflict intensity when the two coalitions differ sufficiently in size and the degree of local interdependencies of conflicts is high. While conventional wisdom suggests that conflict intensity tends to increase in the returns to scale technology, our findings show that the result can be reversed in complete bipartite conflict networks. Such a non-monotonic relationship between conflict intensity and returns to scale technology may offer an explanation for several real-world observations. For instance, although the return from winning a war in modern times is greater than that from an ancient war, the overall conflict investments by opposing countries do not always present an increasing trend. During the period when new sales modes were first introduced, sales staff in large companies took a positive attitude towards such a technological shock. However, one might observe many sales persons being less enthusiastic in promoting sales in recent years, even though sales technology continues to develop. Moreover, in many situations, the total investment in conflict networks (i.e., conflict intensity) is considered a pure cost for the entire society. Our theory thus implies that an increase in the returns to scale technology can be beneficial, in the sense that conflict intensity is reduced.

Our paper is closely related to the literature on strategic conflict analysis. While most of the existing literature examines a single conflict between two or more parties (Anderton and Carter, 2009; Esteban and Ray, 1999; 2008; Garfinkel and Skaperdas, 2007; Konrad, 2009; Schelling, 1960; Smith et al., 2014), we allow conflict parties to simultaneously be involved in several conflicts with different opponents. In addition to Franke and Öztürk (2015)'s work, which we closely follow, there are also a few recent papers on conflict networks. König et al. (2017) study how a network of military alliances and enmities affects the intensity of a conflict from both theoretical and empirical perspectives. Xu et al. (2018) provide a variational inequality approach to study the nexus of conflicts under a general contest technology. Cortes-Corrales and Gorny (2018) investigate agents' behavior in bilateral contests when valuations and efficiencies are heterogenous. Our paper differs from these mainly in two respects. First, we consider a general Tullock conflict network model with returns to scale technology, which takes the lottery contest with constant returns to scale in Franke and Öztürk (2015) as a special case. Second, our main purpose is to analyze how conflict technology affects the equilibrium behavior of agents, which differs from previous studies.

Our study is also related to the literature on contest design. Earlier literature on contests studies the effect of contest structure on the optimal design (e.g., Epstein et al., 2011; Fu and Lu, 2012; Gershkov et al., 2009; Giebe and Schweinzer, 2014; Gradstein and Konrad, 1999; Lu et al., 2017). A handful of studies allow the dissipation factor to be a primary instrument of the contest designer; see, for example, Che and Gale (1997, 2000), Fang (2002), Nti (2004), Wang (2010), and Fu et al. (2015). While all of these assume contest designer could select the competitive mechanism to maximize effort, we treat the dissipation factor as an exogenous value that measures the conflict technology, as it cannot be controlled by any individual conflict party in a conflict network. Thus, unlike those works, which consider the optimal design problem, we study the effect of the dissipation factor on the aggregate effort level.

The remainder of the paper is organized as follows. In Section 2, we introduce conflict network games. In Section 3, we study the equilibrium behaviors and the effects of returns to scale technology in bipartite conflict network games. Section 4 provides several discussions and extensions, and Section 5 concludes the paper. All technical proofs are collected in Appendix A.

2. Model

We consider a conflict network model, adapted from Franke and Öztürk (2015), by incorporating returns to scale technology. Instead of studying general network structures, we are interested in a particular network structure, i.e., complete bipartite networks. In such networks, there are two opposing coalitions, in which each agent of one coalition is in conflict with all members of the other (hostile) coalition but not in conflict with any other members of the same coalition. A complete bipartite network, which has the special feature of asymmetry between two coalitions, is perhaps the most classical type of network. Consider, for example, wars among countries in two opposing military alliances (e.g., the Allies and the Axis in World War II) and conflicts among members of two different ideologies. Each of these two networks is asymmetric, in the sense that the sizes of two coalitions are different. See Fig. 1(a). In addition, complete bipartite networks also cover as a special case the network in which one representative agent in a coalition is involved in conflicts with multiple agents in the other coalition. One classical example of this is an empire fighting its enemies. Another example is a large company that diversifies its product lines in the face of direct competition with many individual sellers offering different varieties of products. See Fig. 1(b).

In the following, we detail the components of a *(complete) bipartite conflict network game*.

Players. There are two opposing sets of risk-neutral agents (i.e., two coalitions), A and B. Let $N = A \cup B$, $n_A = |A|$, $n_B = |B|$, and $n = |N|$. Without loss of generality, we assume that $0 < n_A \le n_B$, implying that B is a larger coalition.

Conflict network structure. In a complete bipartite conflict network, each member of coalition A is in bilateral conflicts with all members of coalition B, and vice versa.

Conflict technology. We model each bilateral conflict as a contest, which involves one agent in coalition A and one agent in coalition B. The outcome of each contest depends on the simultaneous strategic behaviors (i.e., contest efforts) of the involved agents. For each agent i in coalition A, denote her *effort* in the contest against her rival j in coalition B by $x_{ij} \in \mathbb{R}_+$ (we use $\mathbb{R}_+$ to denote the set of all nonnegative real numbers), her *effort vector* against all of her rivals by an n_B-dimensional vector $\boldsymbol{x}_i = (x_{ij})_{j \in B}$, and her *total effort* against all of her rivals by $X_i = \sum_{j \in B} x_{ij}$. The vector of contest efforts directed against agent i in A by all of her rivals is denoted by $\boldsymbol{x}_{-i} = (x_{ji})_{j \in B}$. The above notations are defined similarly for agents in B.

In each contest, the agent who wins the contest obtains an exogenous prize $V > 0$, and the agent who loses receives nothing.[2] The outcome of each contest is specified by a contest success function following Tullock (1980): in the contest

[2] Franke and Öztürk (2015) model each bilateral conflict as a transfer contest, in which some amount of resources is transferred from the loser to the winner. Our specification is strategically equivalent to their setting if the prize value is scaled up to $2V$.

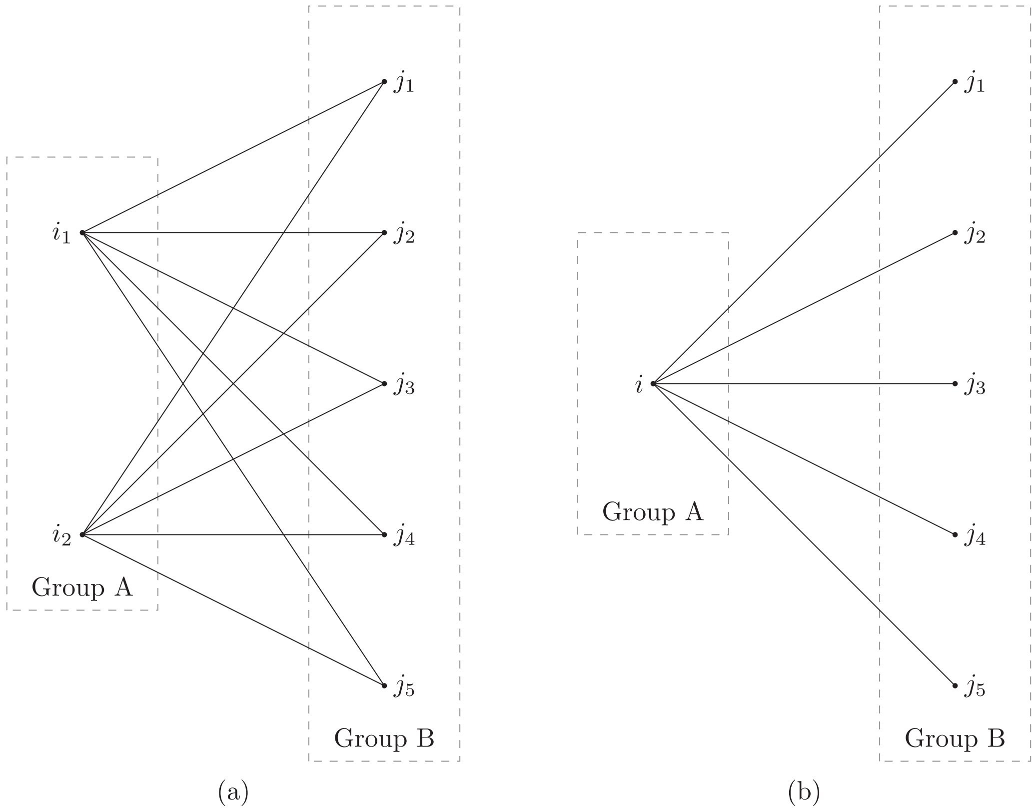

Fig. 1. Complete bipartite network structures.

between agent i and her rival j, when they exert efforts x_{ij} and x_{ji}, respectively, the winning probability of agent i is

$$p(x_{ij}, x_{ji}) = \begin{cases} \frac{x_{ij}^r}{x_{ij}^r + x_{ji}^r}, & \text{if } x_{ij} + x_{ji} > 0, \\ \frac{1}{2}, & \text{if } x_{ij} = x_{ji} = 0, \end{cases}$$

where r is a positive constant. The parameter r represents the *returns to scale technology in effort spending*: if $r > 1$ (resp. $r = 1$ or $r < 1$), the returns to scale are increasing (resp. constant or decreasing). Throughout the paper, we assume that $r \leq 1$ unless otherwise stated. This ensures the existence and uniqueness of a pure-strategy equilibrium in our model.[3]

Valuation, cost, and payoff. For each agent i, exerting effort against her rivals induces a cost $C(X_i)$, which is a function of the agent's total effort in all conflicts. For the ease of presentation, we assume that $C(X_i) = \frac{1}{1+\alpha}(X_i)^{1+\alpha}$, where α is a positive constant.[4] This cost function captures the local interdependencies of conflicts in networks—an agent's behaviors in distinct bilateral conflicts are related. Specifically, an agent's effort making in one bilateral conflict would impose a negative externality on her effort elicited in other conflicts, i.e., a higher effort in one bilateral conflict leads to a higher marginal cost of effort in any other bilateral conflict. Intuitively, such a negative externality would become stronger if conflicts are more interdependent. In addition, the parameter α in the cost function measures the degree of local interdependencies of conflicts: a higher α means a higher local interdependency, which implies a stronger externality among different conflicts. Therefore, α is a key parameter that reflects the special feature of conflict networks in which local interdependencies of conflicts exist. At the extreme, when $\alpha = 0$, the conflict network model becomes identical to a traditional contest model with several independent conflicts.

The expected payoff of each agent is assumed to be additively separable in the costs and expected prizes of all involved contests. In the conflict network, when the agents' strategy profile is $\boldsymbol{x} = (\boldsymbol{x}_k)_{k \in N}$, the expected payoff of agent i in coalition A is

$$u_i(\boldsymbol{x}) = u_i(\boldsymbol{x}_i, \boldsymbol{x}_{-i}) = V \cdot \sum_{j \in B} p(x_{ij}, x_{ji}) - C(X_i),$$

[3] In Section 4.3, we discuss the case with increasing returns to scale, where a symmetric pure-strategy equilibrium may fail to exist.

[4] As in Franke and Öztürk (2015), we assume that the cost function is the same for all agents. We show in an extension that our main results continue to hold when allowing for cost asymmetry between agents; see Section 4.1. We also show that our main results of the effect of returns to scale technology on equilibrium efforts continue to hold for a general cost function $C(\cdot)$; see Section 4.2.

and the expected payoff of agent j in coalition B is

$$u_j(\boldsymbol{x}) = u_j(\boldsymbol{x}_j, \boldsymbol{x}_{-j}) = V \cdot \sum_{i \in A} p(x_{ji}, x_{ij}) - C(X_j).$$

3. Equilibrium analysis

In this section, we first characterize the unique pure-strategy Nash equilibrium in the conflict network model and then analyze how agents' equilibrium behaviors can be affected by returns to scale technology.

3.1. Equilibrium characterization

Define the size ratio of the two coalitions as $\theta = \frac{n_A}{n_B}$, where $\theta \in (0, 1]$. We also let $\theta_\alpha = \theta^{\frac{\alpha}{1+\alpha}}$, which measures the "adjusted" size ratio of the two coalitions, taking into account the degree of local interdependencies of conflicts α. Notice that $\theta_\alpha \in (0, 1]$ given $\alpha > 0$. Moreover, both θ and θ_α measure the asymmetry between the two coalitions: while θ directly captures the asymmetry in coalition sizes, θ_α indirectly measures the relative group size for a certain degree of local interdependencies of conflicts. Intuitively, as α increases, the marginal cost of an agent in the small coalition gets much higher than that of an agent in the large coalition. This implies that each agent in the small coalition becomes weaker than any of her rivals in the large coalition. In summary, a higher θ (or a higher θ_α for any given α) implies a weaker asymmetry between the two coalitions.

Some of our equilibrium analysis follows directly from Franke and Öztürk (2015). First, the results on the existence and uniqueness of an equilibrium in Proposition 1 of Franke and Öztürk (2015) continue to hold in our model. Moreover, Lemma 1 in Franke and Öztürk (2015) proves that the equilibrium is interior, where the winning probability function takes the form of Tullock with $r = 1$. Similarly, we can show that the interior result still holds even for $0 < r < 1$. The details are given in Section A.1.

Therefore, the unique interior pure-strategy equilibrium $\boldsymbol{x}^* = (\boldsymbol{x}_k^*)_{k \in N}$ is characterized by the following first-order conditions:

$$V \cdot \frac{r(x_{ij}^*)^{r-1}(x_{ji}^*)^r}{[(x_{ij}^*)^r + (x_{ji}^*)^r]^2} - (X_i^*)^\alpha = 0 \text{ and } V \cdot \frac{r(x_{ji}^*)^{r-1}(x_{ji}^*)^r}{[(x_{ji}^*)^r + (x_{ij}^*)^r]^2} - (X_j^*)^\alpha = 0, \tag{1}$$

for each agent $i \in A$ and each agent $j \in B$. Here, $\boldsymbol{x}_i^* = (x_{ij}^*)_{j \in B}$, $X_i^* = \sum_{j \in B} x_{ij}^*$, $\boldsymbol{x}_j^* = (x_{ji}^*)_{i \in A}$, and $X_j^* = \sum_{i \in A} x_{ji}^*$. It can be derived directly that $x_{ij}^* = x_a$ and $x_{ji}^* = x_b$ for each agent $i \in A$ and each agent $j \in B$, where

$$x_a = \left[\frac{Vr\theta_\alpha^r}{n_B^\alpha (1+\theta_\alpha^r)^2}\right]^{\frac{1}{1+\alpha}} \text{ and } x_b = \left[\frac{Vr\theta_\alpha^r}{n_A^\alpha (1+\theta_\alpha^r)^2}\right]^{\frac{1}{1+\alpha}}. \tag{2}$$

The following proposition formally summarizes the above equilibrium result.

Proposition 1. *In a complete bipartite conflict network game, there exists a unique pure-strategy equilibrium, in which the equilibrium effort of each agent in coalition A against each agent in B is x_a, and the equilibrium effort of each agent in coalition B against each agent in A is x_b, where x_a and x_b are given by Eq. (2).*

Based on the uniqueness result in Proposition 1, the equilibrium behavior of each agent can be further studied. When the two coalitions are of equal size (i.e., $n_A = n_B$), each agent would behave symmetrically by exerting the same level of effort in every bilateral conflict and incurring the same amount of total effort. The more interesting case, which we focus on hereafter, is when both coalitions are asymmetric (i.e., $n_A < n_B$), so that agents in different coalitions behave differently.

Comparing the equilibrium behavior of agents in different coalitions, we find that each member of the larger coalition exerts more equilibrium effort in each bilateral conflict but less total equilibrium effort (see Section A.1 for details). This reflects the fact that when competing with fewer rivals, each agent in the larger coalition competes more aggressively (because of the lower effective cost against each rival) but incurs a lower total effort (because there are fewer bilateral conflicts involved). Moreover, each agent in the larger coalition also wins each bilateral conflict with a higher probability and receives a higher equilibrium payoff. As a result, the larger coalition, as a whole, exerts less total effort and obtains a higher total equilibrium payoff. All these results are qualitatively consistent with the findings in Franke and Öztürk (2015) where $r = 1$.

We further consider how the size of each coalition affects the equilibrium behaviors of agents. The results are summarized as follows: first, the equilibrium effort of each small-coalition agent against each of her rivals increases in the size of the smaller coalition and decreases in the size of the larger coalition, while the equilibrium effort of each large-coalition agent against each of her rivals decreases in the size of each coalition. Second, the total equilibrium effort of each small-coalition agent increases in the size of each coalition, while the total equilibrium effort of each large-coalition agent increases in the size of the smaller coalition and decreases in the size of the larger coalition. Third, the total equilibrium effort of each coalition and the conflict intensity increase in the size of each coalition. The details are given in Section A.1.

Intuitively, as the size of the smaller coalition increases, each large-coalition agent has less incentive to exert effort against any of her rivals (since she must incur a higher marginal cost in each bilateral conflict when competing with more rivals) but exerts higher total effort (since she is involved in more bilateral conflicts). Similarly, an increase in the size of the larger coalition would lower the effort level of each small-coalition agent, resulting in lower effort of each large-coalition agent. This logic, however, does not exactly work for agents in the smaller coalition who are also affected by a disincentive effect: in networks where two coalitions are asymmetric, agents in the smaller coalition tend to have less incentive to exert effort. As the size of the smaller coalition increases, this disincentive effect is weakened, which implies that agents in the smaller coalition would compete more aggressively. This explains why the equilibrium effort of each small-coalition agent against each of her rivals increases, rather than decreases, in the size of the smaller coalition. The relationship between the total effort of each small-coalition agent and coalition sizes can also be explained following a similar argument.

The fact that both the total effort of each coalition and overall conflict intensity increase in the size of each coalition suggests a positive relationship between the total effort and the number of bilateral conflicts in complete bipartite network games. This result is a generalization of the findings of Franke and Öztürk (2015), whose model focuses on a special conflict technology with constant returns to scale.

3.2. Returns to scale technology

The most important feature of our network model is the introduction of the returns to scale technology, which measures the competitiveness of each bilateral conflict. Although the conflict technology of a network seems exogenous, it may change over time (e.g., from ancient war to high-tech war or from physical store sales to e-commerce sales) or vary across different networks (e.g., global war/competition vs. country-level war/competition). An interesting question is how the conflict technology affects the equilibrium behavior of agents. Specifically, our aim is to address how a change in the returns to scale technology affects the equilibrium effort of each agent, total equilibrium effort of each coalition, and the conflict intensity.

In the literature on contests without network structures, it is widely acknowledged that as the contest technology parameter r increases, a contest becomes more discriminatory such that it tends to elicit higher individual effort and total effort. One might expect a similar result to hold in conflict networks. However, we find that such a result does not necessarily hold in complete bipartite conflict network games.

Let $\bar{\theta}_\alpha$ denote a critical "adjusted" size ratio of two coalitions, which is the unique solution to the equation $1+\frac{1-\theta_\alpha}{1+\theta_\alpha}\ln\theta_\alpha=0$. Note that $\bar{\theta}_\alpha\approx 0.2137$. The following proposition summarizes the impact of the returns to scale technology on the equilibrium efforts of agents in complete bipartite conflict networks.

Proposition 2. *In a complete bipartite network game, we have the following properties.*

1. *If the two coalitions are sufficiently asymmetric ($\theta_\alpha<\bar{\theta}_\alpha$), the equilibrium effort of each agent in any bilateral conflict, the total equilibrium effort of each agent, the total equilibrium effort of each coalition, and conflict intensity are all increasing in r when $r\in(0,\bar{r})$ and decreasing in r when $r\in(\bar{r},1]$, where $\bar{r}\in(0,1)$ is the unique solution to $\frac{1}{r}+\frac{1-\theta_\alpha^r}{1+\theta_\alpha^r}\ln\theta_\alpha=0$ for a given θ_α.*
2. *If the two coalitions are not sufficiently asymmetric ($\bar{\theta}_\alpha\le\theta_\alpha\le 1$), the equilibrium effort of each agent in any bilateral conflict, the total effort of each agent, the total investment of each coalition, and conflict intensity are all monotonically increasing in r when $r\in(0,1]$.*

Proof. See the Appendix (Section A.2). □

Proposition 2 shows that the effects of returns to scale technology on equilibrium efforts crucially depend on the relative asymmetry of the two coalitions θ_α. When the two coalitions are not sufficiently asymmetric (i.e., $\theta_\alpha\ge\bar{\theta}_\alpha$), which occurs when the size ratio of the two coalitions is large or when the degree of local interdependencies of conflicts is low, equilibrium efforts are always increasing in r. However, when the two coalitions are sufficiently asymmetric (i.e., $\theta_\alpha<\bar{\theta}_\alpha$), there exists an inverted U relationship between equilibrium efforts and returns to scale technology.

Intuitively, two effects can be identified: a *competition effect* and a *discouragement effect*. First, as returns to scale technology increases, the competition between agents in each bilateral conflict becomes fiercer, which implies that each agent has stronger incentives to exert effort (competition effect). Second, a small-coalition agent realizes that she is less likely to win in each conflict as r increases. That is, in a symmetric equilibrium, if she wants to win with a higher probability in each conflict, she has to exert more effort in each competition, which leads to a huge effort cost since she faces many rivals. This discourages her from investing more. In response, an agent in the larger coalition may want to lower her own effort level (discouragement effect). Since these two effects work in opposite directions, the net effect on the equilibrium efforts would depend on which one dominates. When the two coalitions are not sufficiently asymmetric, the discouragement effect is always dominated by the competition effect, which leads to a usually positive relationship between r and equilibrium efforts. By contrast, when the two coalitions are sufficiently asymmetric, which occurs when the size ratio of the two coalitions is small and the degree of local interdependencies of conflicts is high, either effect can dominate: the discouragement effect continues to be dominated for a low level of r but becomes stronger than the competition effect when r reaches some critical level $\bar{r}$. As a result, the equilibrium effort of each agent first increases and then decreases as r increases.

Since the conflict intensity is the aggregation of individual efforts, we also find an inverted U relationship between returns to scale technology and conflict intensity, when the two coalitions are sufficiently asymmetric, i.e., when the size

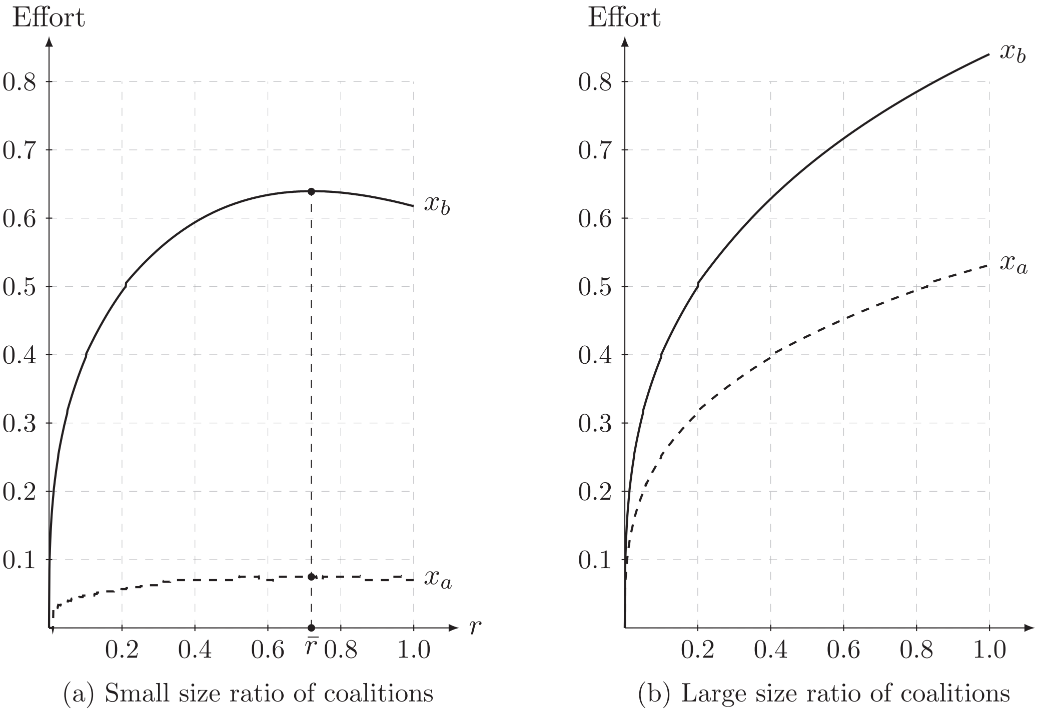

Fig. 2. The relationship between equilibrium efforts and returns to scale technology.

ratio of the two coalitions is small and the degree of local interdependencies of conflicts is high. Otherwise, conflict intensity is strictly increasing in returns to scale technology.

To see how the relationship between equilibrium efforts and returns to scale technology is affected by the relative asymmetry between the two coalitions, we consider two examples in Fig. 2. Fig. 2(a) depicts the case with $V = 10$, $\alpha = 2$, $n_A = 2$, and $n_B = 50$, which implies that $\theta = \frac{1}{25}$ and $\theta_\alpha = \left(\frac{1}{25}\right)^{\frac{2}{3}}$. This captures the case in which the two coalitions are sufficiently different in size. In this case, the critical value $\bar{r} = 0.7192$. It is clear that the equilibrium effort of each agent (x_a or x_b) increases in r when $r < \bar{r}$, reaches its maximum when r reaches the critical level $\bar{r}$ and decreases thereafter. Fig. 2(b) depicts the case with $V = 10$, $\alpha = 2$, $n_A = 2$, and $n_B = 4$, where $\theta = \frac{1}{2}$ and $\theta_\alpha = (\frac{1}{2})^{\frac{2}{3}}$, so that the two coalitions are close in size. It shows a positive relationship between equilibrium efforts (x_a and x_b) and the returns to scale technology parameter r for $r \in (0, 1]$.

Proposition 2 can be used to understand the behaviors of agents in conflict networks. Contrary to the conventional wisdom that conflict intensity tends to be higher as the returns to scale technology increases, our finding suggests that the result can be reversed in complete bipartite networks in which two coalitions are sufficiently different in size. This finding offers a possible explanation for some phenomena observed in real-world conflict networks. Although the return from winning a war in modern times is higher than that from an ancient war, opposing military alliances do not always expand their military expenditures. The non-monotonic relationship between the returns to scale technology and effort spending of agents seems to be more relevant in the case of market competition. With the development of the internet and modern technology, online sales have had a considerable impact on the nature of competition among sellers and on consumers' daily life. During the period when internet sales were first introduced, intensified competition between traditional brick-and-mortar firms and individual online sellers led to huge investments in sales promotion. In recent years, however, both sales staff working in a shopping mall and online sellers seem much less enthusiastic about promoting their products. According to our result, such a change in sellers' attitudes towards marketing products occurs because of “the discouragement effect”, which results in salespersons exerting less effort even if market competition is fiercer.

Our result on the relationship between equilibrium efforts (and conflict intensity) and returns to scale technology in complete bipartite conflict network games is similar to the findings in the Tullock contest literature. In a Tullock contest with two asymmetric players, Wang (2010) shows that provided that a pure-strategy equilibrium exists, the equilibrium effort of each player initially increases and then falls as r increases when the asymmetry is sufficiently large. One might immediately conjecture that the non-monotonic relationship between equilibrium efforts and returns to scale technology as established in these two frameworks is a result of the heterogeneity of players. However, we find that this is not sufficient to ensure that the non-monotonic result holds in our conflict network model. In Wang (2010), the difference in players' costs is the only source of the asymmetry between players. In our model, however, the asymmetry between players is determined by two factors: the size ratio of coalitions and the degree of local interdependencies of conflicts. Without of the local interdependencies of conflicts (i.e., $\alpha = 0$), it is straightforward to see that the equilibrium efforts of agents are always increasing in returns to scale technology, even if the two coalitions are sufficiently different in size. As a result, the inverted U relationship between equilibrium efforts and returns to scale technology is non-trivial in our conflict network model, in which local interdependencies of conflicts play an important role.

Comparative statics results on the critical value $\bar{r}$. Proposition 2 suggests that the critical value $\bar{r}$ plays an important role in capturing the relationship between equilibrium efforts of agents and returns to scale technology. In fact, the critical value corresponds to the level of returns to scale under which the equilibrium efforts of each agent and conflict intensity are maximized. If the conflict intensity is regarded as a pure cost for the whole society, as in many conflict networks, the critical value $\bar{r}$ exactly captures the situation in which such an inefficient social cost is the highest.

One might wonder how the critical value is affected by different factors. Given θ_α, recall that $\bar{r}$ is the unique solution to the equation

$$\frac{1}{r} + \frac{1-\theta_\alpha^r}{1+\theta_\alpha^r}\ln\theta_\alpha = 0.$$

Thus, it is apparent that $\bar{r}$ is determined by the adjusted size ratio θ_α, which depends on both the size ratio θ and the degree of local interdependencies of conflicts α. From the proof of Proposition 2, we know that $\bar{r}$ is increasing in θ_α. Note further that θ_α is increasing in θ but decreasing in α, which follows from

$$\frac{\partial\theta_\alpha}{\partial\theta} = \frac{\alpha}{1+\alpha}\theta^{-\frac{1}{1+\alpha}} > 0 \text{ and } \frac{\partial\theta_\alpha}{\partial\alpha} = \frac{1}{(1+\alpha)^2}\theta_\alpha\ln\theta < 0.$$

Therefore, we obtain that $\frac{\partial\bar{r}}{\partial\theta} > 0$ and $\frac{\partial\bar{r}}{\partial\alpha} < 0$. The following result formally summarizes how the critical value $\bar{r}$ is affected by these two factors.

Proposition 3. *The critical level of returns to scale $\bar{r}$ increases in the size ratio of the coalitions (θ) and decreases in the degree of local interdependencies of conflicts (α).*

Intuitively, as the discouragement effect increases in the relative asymmetry between the two coalitions, the critical level of returns to scale $\bar{r}$ decreases in the relative asymmetry between the two coalitions θ_α, which also depends on the size ratio of the two coalitions θ and the degree of local interdependencies of conflicts α. On the one hand, when the size ratio of coalitions increases, the discouragement effect is weakened, which causes all agents to exert more effort. On the other hand, when the degree of local interdependencies of conflicts becomes stronger, the marginal cost of an agent in each bilateral conflict increases, especially for an agent in the smaller coalition who is involved in more conflicts. As a result, such an agent starts to compete less aggressively than it had previously. In response, an agent in the larger coalition also slacks off under a low level of returns to scale.

The following two figures explicitly show how the changes in θ and α affect $\bar{r}$.

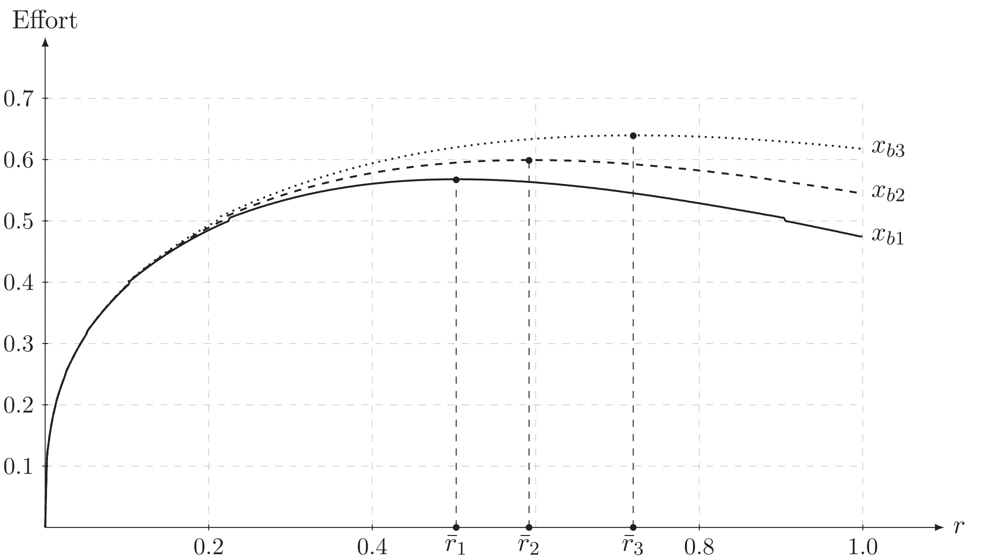

Fig. 3. The effect of size ratio on the critical level of returns to scale.

Fig. 3 shows a positive relationship between $\bar{r}$ and θ. The solid curve, the dashed curve, and the dotted curve depict the case with $V=10$, $\alpha=2$, $n_A=2$ and $n_B=200$ (case 1); the case with $V=10$, $\alpha=2$, $n_A=2$, and $n_B=100$ (case 2); and the case with $V=10$, $\alpha=2$, $n_A=2$, and $n_B=50$ (case 3), respectively. The size ratios in these three cases are $\frac{1}{100}$ (case 1), $\frac{1}{50}$ (case 2), and $\frac{1}{25}$ (case 3), respectively. Simple calculation shows that the critical values in these three cases are $\bar{r}_1=0.5027$, $\bar{r}_2=0.5918$, and $\bar{r}_3=0.7192$, respectively. Therefore, as the size ratio increases from case 1 to case 3, the critical level also grows. The effect of group size on the critical value could also explain why the impact of a technological change may vary across markets. Although the introduction of new sales modes has similar effects on competition in both local areas and at the national level, large companies that sell products in a larger region and face more competitors seem to experience greater fluctuations in their sales revenues than do some dominant sellers in localized regions.

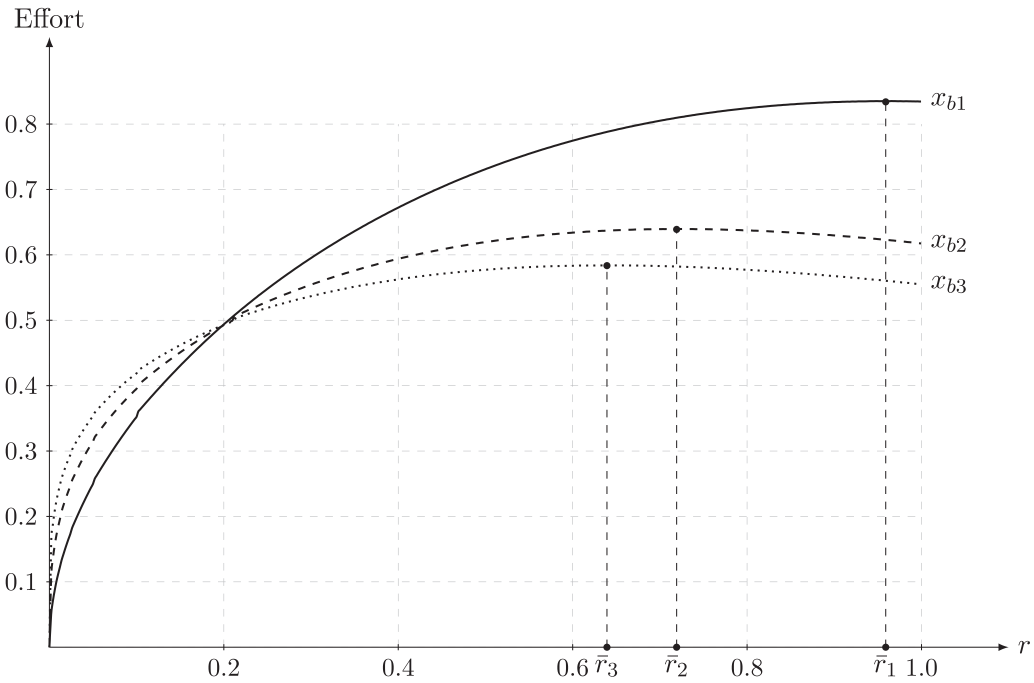

Fig. 4. The effect of the degree of local interdependencies on the critical level of returns to scale.

By contrast, Fig. 4 shows a negative relationship between $\bar{r}$ and α. In Fig. 4, the solid curve, the dashed curve, and the dotted curve depict the case with $V=10$, $\alpha=1$, $n_A=2$, and $n_B=50$ (case 1); the case with $V=10$, $\alpha=2$, $n_A=2$, and $n_B=50$ (case 2); and the case with $V=10$, $\alpha=3$, $n_A=2$, and $n_B=50$ (case 3), respectively. The critical values in these three cases are $\bar{r}_1=0.9590$, $\bar{r}_2=0.7192$, and $\bar{r}_3=0.6393$, respectively. Hence, the critical value decreases in α. Our results suggest that returns to scale technology is more likely to have a negative impact on agents' total investment expenditures in networks with stronger local interdependencies between distinct conflicts. For instance, firms in manufacturing industries belong to networks with stronger local interdependencies than do those in retailing industries, because the former are more capacity constrained so that the marginal cost in one product category is more affected by the resources spent on other products. According to our result, a positive technological shock tends to have a larger (likely negative) impact on production in manufacturing industries.

4. Discussions and extensions

In this section, we extend the benchmark model in several directions. First, we consider the situation in which agents in different coalitions may have different cost functions. Second, we extend our model by allowing a more general cost function. Third, we explain why we focus on $r \le 1$ and discuss how our main results can be extended by allowing r to be larger than one. The detailed workings of each part are relegated to the Appendix.

4.1. Cost asymmetry

We have thus far focused on the case in which each agent has the same cost function. In this section, we explore the possibility that agents in different coalitions may differ in their investment costs. Suppose that each agent i in coalition A has a cost function of $C_A(X_i)=\frac{\beta}{1+\alpha}X_i^{1+\alpha}$, while each agent j in coalition B has a cost function of $C_B(X_j)=\frac{1}{1+\alpha}X_j^{1+\alpha}$, where

$\beta > 0$. Thus, all agents in the same coalition are still assumed to have the same cost function, which differs from that of agents in the other coalition. The parameter β measures the cost asymmetry between coalitions. When $\beta > 1$ (resp. $\beta < 1$), agents in the smaller coalition incur a higher (resp. lower) investment cost.

Despite the presence of cost asymmetry, the existence and uniqueness of a pure-strategy equilibrium, which is also interior, continue to hold. The equilibrium $\boldsymbol{x}^* = (\boldsymbol{x}_k^*)_{k\in N}$ can be characterized by the following first-order conditions

$$V \cdot \frac{r(x_{ij}^*)^{r-1}(x_{ji}^*)^r}{[(x_{ij}^*)^r + (x_{ji}^*)^r]^2} - \beta (X_i^*)^\alpha = 0 \text{ and } V \cdot \frac{r(x_{ji}^*)^{r-1}(x_{ji}^*)^r}{[(x_{ji}^*)^r + (x_{ij}^*)^r]^2} - (X_j^*)^\alpha = 0,$$

for all players $i\in A$ and $j\in B$. Here, $\boldsymbol{x}_i^* = (x_{ij}^*)_{j\in B}$, $X_i^* = \sum_{j\in B} x_{ij}^*$, $\boldsymbol{x}_j^* = (x_{ji}^*)_{i\in A}$, and $X_j^* = \sum_{i\in A} x_{ji}^*$. Simultaneously solving these equations implies that the equilibrium is again symmetric, that is, $x_{ij}^* = x'_a$, $x_{ji}^* = x'_b$, and x'_a and x'_b solve

$$Vr \cdot \frac{(x'_a)^r (x'_b)^r}{[(x'_a)^r + (x'_b)^r]^2} = \beta n_B^\alpha (x'_a)^{1+\alpha},$$
$$Vr \cdot \frac{(x'_b)^r (x'_a)^r}{[(x'_b)^r + (x'_a)^r]^2} = n_A^\alpha (x'_b)^{1+\alpha}.$$

Therefore, we obtain

$$x'_a = \left[\frac{Vr(\theta'_\alpha)^r}{\beta n_B^\alpha (1 + (\theta'_\alpha)^r)^2}\right]^{\frac{1}{1+\alpha}} \text{ and } x'_b = \left[\frac{Vr(\theta'_\alpha)^r}{n_A^\alpha (1 + (\theta'_\alpha)^r)^2}\right]^{\frac{1}{1+\alpha}},$$

where $\theta'_\alpha = \theta_\alpha \beta^{-\frac{1}{1+\alpha}}$. Note that $\theta'_\alpha \leq 1$ if and only if $\beta \geq \theta^\alpha$. The equilibrium outcomes now depend on the cost asymmetry β.

We first consider the case in which $\beta \geq \theta^\alpha$, which means that the agents in the small coalition have higher costs (i.e., $\beta > 1$) or slightly lower costs (i.e., $1 \geq \beta \geq \theta^\alpha$). We find that each small-coalition agent again exerts lower effort than any of her rivals in each bilateral conflict (i.e., $x'_a < x'_b$) but may also exert a lower level of total effort (i.e., $X_a < X_b$) if the cost disadvantage is sufficiently strong (i.e., $\beta > \theta^{-1}$). Most importantly, Proposition 2 continues to hold once we replace θ_α in that proposition with θ'_α. In other words, there also exists an inverted U relationship between the returns to scale technology and (individual and total) equilibrium efforts when two coalitions are sufficiently different in size (i.e., $\theta'_\alpha < \bar{\theta}_\alpha$). Proposition 3 also holds, except that when $\beta > \theta^{-1}$, the relationship between the critical value $\bar{r}$ and α becomes positive. In addition, since θ'_α is decreasing in β, we further establish a negative relationship between $\bar{r}$ and β. This implies that as the cost of each small-coalition agent larger, it is more likely that equilibrium efforts change non-monotonically as the returns to scale technology increases.

Next we consider the case in which $\beta < \theta^\alpha$ where agents in the small coalition have a cost advantage over their rivals in the large coalition. If we define $\theta_\alpha^{inv} = \frac{1}{\theta'_\alpha} \in (0, 1)$, then the equilibrium efforts can be rewritten as

$$x'_a = \left[\frac{Vr(\theta_\alpha^{inv})^r}{\beta n_B^\alpha \left(1 + (\theta_\alpha^{inv})^r\right)^2}\right]^{\frac{1}{1+\alpha}} \text{ and } x'_b = \left[\frac{Vr(\theta_\alpha^{inv})^r}{n_A^\alpha \left(1 + (\theta_\alpha^{inv})^r\right)^2}\right]^{\frac{1}{1+\alpha}}.$$

Contrary to the results in Section 3.1, each agent in the small coalition now exerts higher effort than any of her rivals in each bilateral conflict (i.e., $x'_a > x'_b$) and a higher level of total effort (i.e., $X_a > X_b$). This suggests that a small-coalition agent now acts as a stronger player compared to each rival in the large coalition. However, the main results in Proposition 2 continue to hold once we replace θ_α with θ_α^{inv}. All results in Proposition 3 are reversed, reflecting the feature that the cost asymmetry between agents in different coalitions increases when agents in the small coalition have lower costs or the group size ratio is higher, both of which strengthen the cost advantage of small-coalition agents.

4.2. General cost function

Our main results are derived based on a specific cost function, which is a power function. However, we show that our main result of the effect of returns to scale technology on equilibrium efforts continues to hold for a general cost function $C(X_i)$. Consider a general cost function $C(\cdot)$, which is strictly increasing and convex, i.e., $C'(\cdot) > 0$, $C''(\cdot) > 0$, and $C(0) = 0$.

In a (symmetric) pure-strategy equilibrium, we have $x_{ij}^* = x_a$ and $x_{ji}^* = x_b$ for each agent $i\in A$ and each agent $j\in B$. Then, x_a and x_b solve the following first-order conditions:

$$Vr \cdot \frac{x_a^r x_b^r}{(x_a^r + x_b^r)^2} = C'(X_a) x_a \text{ and } Vr \cdot \frac{x_a^r x_b^r}{(x_a^r + x_b^r)^2} = C'(X_b) x_b,$$

where $X_a = n_B x_a$ and $X_b = n_A x_b$. The above equations also imply that

$$C'(X_a) x_a = C'(X_b) x_b. \tag{3}$$

This establishes an increasing relationship between x_a and x_b, given that $C'(X_i) x_i$ is increasing in x_i for $i = a, b$.

We further define a function $g(x_b) = \frac{x_a}{x_b}$, where x_a is an implicit function of x_b determined by Eq. (3). To show that there exists an inverted U relationship between equilibrium efforts (x_a and x_b) and r, we find that it is sufficient to prove $g'(x_b) \le 0$. The details are provided in Section A.3. To guarantee $g'(x_b) \le 0$, we present a sufficient condition:

$$h'(x) \ge 0, \tag{4}$$

where $h(x) = \frac{C''(x)}{C'(x)}x$. That is, the cost function is sufficiently convex. Note that when $C(x) = \frac{1}{1+\alpha}x^{1+\alpha}$, we have $h(x) = \frac{1}{\alpha}$, where Eq. (4) holds.

Thus, given that condition (4) holds (or $g'(x_b) \le 0$), there is a non-monotonic relationship between the returns to scale technology and equilibrium efforts. In particular, the equilibrium efforts x_b and x_a are increasing in r when r is small and decreasing in r when r is sufficiently large. Therefore, our main result of the effect of return to scale technology on equilibrium efforts continues to hold for a general cost function.

4.3. Increasing returns to scale

We have thus far assumed non-increasing returns to scale (i.e., $r \le 1$), which ensures the existence and uniqueness of a pure-strategy equilibrium in our model. Once we allow for increasing returns to scale (i.e., r to be larger than one), the payoff function of each agent is no longer concave. It remains an open question to establish the existence and uniqueness of equilibrium in conflict network games where players' payoff functions are not concave. In fact, in complete bipartite conflict network games, the symmetric pure-strategy equilibrium, which we have characterized, may fail to exist.

For a symmetric pure-strategy equilibrium to exist, two set of conditions must hold. First, the second-order (necessary) conditions must be satisfied for each agent in equilibrium. Second, each agent's participation constraint should also be satisfied. It can be derived that the second-order conditions are satisfied for all agents if and only if $r < r_1$, where $r_1 > 1$ is the unique solution to $\psi_1(r) = \theta_\alpha^r - \frac{r-1}{1+r} = 0$. In addition, the participation constraints hold for all agents if and only if $r \le r_2$, where $r_2 > 1$ is the unique solution to $\psi_2(r) = \frac{r}{1+\theta_\alpha^r} - (1+\alpha) = 0$. Therefore, the symmetric equilibrium exists only if $r < r_1$ and $r \le r_2$. Note that both r_1 and r_2 are increasing in θ_α. When θ_α is sufficiently small (i.e., the group size difference is sufficiently large and/or the degree of local interdependencies is sufficiently strong), r cannot be too large for guaranteeing the existence of a symmetric equilibrium. In particular, when θ_α is sufficiently close to zero, the threshold r_1 will also approach 1, which implies that the symmetric equilibrium is unlikely to exist when $r > 1$. The details are given in Section A.4.

Our main result that there exists a non-monotonic relationship between equilibrium efforts and returns to scale technology, however, would continue to apply in the case when $r > 1$, provided the existence of the symmetric pure-strategy equilibrium, which is characterized in Proposition 1.

5. Concluding remarks

Our paper is among the first to study how agents behave in conflict networks with returns to scale technology. Focusing on complete bipartite conflict network games, we find that returns to scale technology plays an important role in affecting the equilibrium behavior of agents. We further identify two important factors that determine the relationship between the equilibrium efforts of agents and returns to scale technology: the size ratio of the two coalitions and the degree of local interdependencies of conflicts. We show that there exists an inverted U relationship between equilibrium efforts and the returns to scale technology, provided that the two coalitions are sufficiently different in size and the degree of local interdependencies of conflicts is high. Such a result offers a possible explanation for why conflict intensity can be reduced as conflict technology varies.

Complete bipartite conflict network games are featured by their asymmetry, which seems critical for the existence of the non-monotonic relationship between equilibrium efforts and returns to scale technology. An obvious idea for further research would be to extend the current model to more general network structures. One might want to identify other factors that also contribute to the asymmetry of agents. It would be interesting to investigate whether an inverted U relationship between equilibrium efforts and returns to scale technology continues to exist in those cases.

Our results can also be tested empirically or by laboratory experiments. For instance, it is possible to use the military expenditure data from opposing military alliances or the data from the sales industry to empirically verify this predicted inverted U relationship. Moreover, it would be intriguing to design contest experiments using the network structure of this study.

Appendix A

A1. Proofs of results in Section 3.1

In this section, we provide the detailed analysis for the results in Section 3.1.

The interiority result is established in the following lemma.

Lemma 1. *The unique pure-strategy Nash equilibrium in a bipartite conflict network game is interior.*

Proof. The proof consists of the following two claims:

Claim 1: Two direct rivals cannot exert zero effort in equilibrium in their respective bilateral conflicts. Consider an arbitrary strategy profile $\boldsymbol{x} = (\boldsymbol{x}_i)_{i\in N}$, where $x_{ij} = x_{ji} = 0$ for some agents $i \in A$ and $j \in B$. Consider another strategy for agent i: $\boldsymbol{x}'_i = (x_{i1}, \ldots, x'_{ij}, \ldots, x_{in_b})$, where $x'_{ij} = \epsilon$ for some sufficiently small constant $\epsilon > 0$. Since $p(x'_{ij}, x_{ji}) = p(\epsilon, 0) = 1 > \frac{1}{2} = p(x_{ij}, x_{ji})$ and $\lim_{\epsilon\to 0} c(X'_i) = c(X_i)$, we have that $u_i(\boldsymbol{x}'_i, \boldsymbol{x}_{-i}) > u_i(\boldsymbol{x}_i, \boldsymbol{x}_{-i})$. That is, $\boldsymbol{x}'_i$ is a profitable deviation of $\boldsymbol{x}_i$.

Claim 2: An agent cannot exert zero effort in equilibrium against a rival with positive conflict investment. Assume by contradiction that there exists an equilibrium $\boldsymbol{x}^* = (\boldsymbol{x}^*_1, \ldots, \boldsymbol{x}^*_i, \ldots, \boldsymbol{x}^*_j, \ldots, \boldsymbol{x}^*_n)$, where $i \in A$, $j \in B$, $x^*_{ij} = 0$, and $x^*_{ji} > 0$. Consider another strategy for agent j: $\boldsymbol{x}'_j = (x_{j1}, \ldots, x'_{ji}, \ldots, x_{jn_a})$, where $x'_{ji} = \frac{1}{2}x_{ji}$, i.e., agent j only reduces the effort exerted against rival i without altering efforts in all other conflicts. Since $p(x'_{ji}, x^*_{ij}) = 1 = p(x^*_{ji}, x^*_{ij})$ and $c(X'_j) < c(X^*_j)$, we have $u_j(\boldsymbol{x}'_j, \boldsymbol{x}^*_{-j}) > u_i(\boldsymbol{x}^*_j, \boldsymbol{x}^*_{-j})$. Hence, $\boldsymbol{x}^*$ cannot be an equilibrium.

Therefore, the unique pure-strategy Nash equilibrium is interior. □

We then compare the equilibrium individual efforts and total efforts between two coalitions. Denote the total equilibrium effort of each agent in coalition A (resp. B) by X_a (resp. X_b), and the total equilibrium effort of all agents in coalition A (resp. B) by X_A (resp. X_B). By symmetry, we have $X_a = n_B x_a$, $X_b = n_A x_b$, $X_A = n_A n_B x_a$, and $X_B = n_A n_B x_b$. Furthermore, the *overall conflict intensity* (or the total equilibrium effort of all agents) in the network, denoted by $\hat{X}$, can be written as $\hat{X} = X_A + X_B$.

Comparing equilibrium effort of each agent in two coalitions, we have

$$\tfrac{x_a}{x_b} = \theta_\alpha < 1,\ \tfrac{X_a}{X_b} = \theta^{-\frac{1}{1+\alpha}} > 1,\ \text{and}\ \tfrac{X_A}{X_B} = \tfrac{x_a}{x_b} = \theta_\alpha < 1.$$

When two coalitions are different in sizes, agents in the smaller coalition tend to compete less aggressively than those in the larger coalition. In particular, an agent in the smaller coalition exerts a lower effort than her rival in each bilateral conflict ($x_a < x_b$), and all agents in the smaller coalition also exert less than those in the larger coalition ($X_A < X_B$). However, the total effort made by each agent in the smaller coalition is larger than that made by each agent in the larger coalition ($X_a > X_b$). This captures the fact that each small-coalition agent is involved in more bilateral conflicts compared to any of her rivals in the larger coalition. Furthermore, note that in each bilateral conflict, the equilibrium winning probability of an agent in coalition A is $p_a = \frac{x_a^r}{x_a^r + x_b^r}$, and that of an agent in coalition B is $p_b = \frac{x_b^r}{x_a^r + x_b^r}$. The fact that $x_a < x_b$ would also imply $p_a < \frac{1}{2} < p_b$, so that an agent in the larger coalition wins each bilateral conflict with a higher probability. Provided that equilibrium payoff of each agent is increasing in her winning probability in each bilateral conflict and decreasing in her total effort, an agent in the larger coalition also receives a higher equilibrium payoff.

In addition, comparing total equilibrium efforts between two coalitions, we obtain $\frac{X_A}{X_B} = \theta_\alpha < 1$, so that the larger coalition invests more than the smaller one does. The total equilibrium payoff of all agents in coalition A is

$$U_A = n_A u_a = n_A n_B V p_a - \tfrac{n_A}{1+\alpha} X_a^{1+\alpha},$$

and that of all agents in coalition B is

$$U_B = n_B u_b = n_A n_B V p_b - \tfrac{n_B}{1+\alpha} X_b^{1+\alpha}.$$

Provided that $\frac{X_a}{X_b} = \theta^{-\frac{1}{1+\alpha}}$, it can be checked that $\frac{n_A}{1+\alpha} X_a^{1+\alpha} = \frac{n_B}{1+\alpha} X_b^{1+\alpha}$, which further implies that $U_A < U_B$. As a result, the net surplus of all members in the larger coalition is also higher.

We further investigate how the number of each coalition affects the equilibrium efforts. First consider the effect of n_A on equilibrium efforts x_a and x_b. It is equivalent to see how n_A affects $\ln x_a$ and $\ln x_b$, where

$$\ln x_a = \frac{1}{1+\alpha}\left[\ln(Vr) + \frac{r\alpha}{1+\alpha}\ln\theta - \alpha\ln n_B - 2\ln\left(1 + \theta^{\frac{\alpha r}{1+\alpha}}\right)\right],$$

$$\ln x_b = \frac{1}{1+\alpha}\left[\ln(Vr) + \frac{r\alpha}{1+\alpha}\ln\theta - \alpha\ln n_A - 2\ln\left(1 + \theta^{\frac{\alpha r}{1+\alpha}}\right)\right].$$

Notice that

$$\frac{\partial \ln x_a}{\partial\theta} = \frac{\partial \ln x_b}{\partial\theta} = \frac{r\alpha}{(1+\alpha)^2\theta}\frac{1-\theta_\alpha^r}{1+\theta_\alpha^r} > 0.$$

Then

$$\frac{\partial \ln x_a}{\partial n_A} = \frac{\partial \ln x_a}{\partial\theta}\frac{\partial\theta}{\partial n_A} = \frac{r\alpha}{(1+\alpha)^2 n_A}\frac{1-\theta_\alpha^r}{1+\theta_\alpha^r} > 0,$$

$$\frac{\partial \ln x_b}{\partial n_A} = \frac{\partial \ln x_b}{\partial\theta}\frac{\partial\theta}{\partial n_A} - \frac{\alpha}{(1+\alpha)n_A} = -\frac{\alpha}{(1+\alpha)n_A}\left(1 - \frac{r}{1+\alpha}\frac{1-\theta_\alpha^r}{1+\theta_\alpha^r}\right) < 0.$$

Moreover, we have

$$\frac{\partial \ln X_a}{\partial n_A} = \frac{\partial \ln x_a}{\partial n_A} > 0,$$

$$\frac{\partial \ln X_b}{\partial n_A} = \frac{\partial \ln x_b}{\partial n_A} + \frac{\partial \ln n_A}{\partial n_A} = \frac{1}{(1+\alpha)n_A}\left(1 + \frac{r\alpha}{1+\alpha}\frac{1-\theta_\alpha^r}{1+\theta_\alpha^r}\right) > 0,$$
$$\frac{\partial \ln X_A}{\partial n_A} = \frac{\partial \ln X_a}{\partial n_A} + \frac{\partial \ln n_A}{\partial n_A} > 0,$$
$$\frac{\partial \ln X_B}{\partial n_A} = \frac{\partial \ln X_b}{\partial n_A} > 0.$$

Analogously, we can study the effect of n_B on the corresponding equilibrium efforts, where

$$\frac{\partial \ln x_a}{\partial n_B} = \frac{\partial \ln x_a}{\partial \theta}\frac{\partial \theta}{\partial n_B} - \frac{\alpha}{(1+\alpha)n_B} = -\frac{r\alpha}{(1+\alpha)^2 n_B}\frac{1-\theta_\alpha^r}{1+\theta_\alpha^r} - \frac{\alpha}{(1+\alpha)n_B} < 0,$$
$$\frac{\partial \ln x_b}{\partial n_B} = \frac{\partial \ln x_b}{\partial \theta}\frac{\partial \theta}{\partial n_B} = -\frac{r\alpha}{(1+\alpha)^2 n_B}\frac{1-\theta_\alpha^r}{1+\theta_\alpha^r} < 0.$$

Furthermore,

$$\frac{\partial \ln X_a}{\partial n_B} = \frac{\partial \ln x_a}{\partial n_B} + \frac{\partial \ln n_B}{\partial n_B} = \frac{1}{(1+\alpha)n_B}\left(1 - \frac{r\alpha}{1+\alpha}\frac{1-\theta_\alpha^r}{1+\theta_\alpha^r}\right) > 0,$$
$$\frac{\partial \ln X_b}{\partial n_B} = \frac{\partial \ln x_b}{\partial n_B} < 0,$$
$$\frac{\partial \ln X_A}{\partial n_B} = \frac{\partial \ln X_a}{\partial n_B} > 0,$$
$$\frac{\partial \ln X_B}{\partial n_B} = \frac{\partial \ln X_b}{\partial n_B} + \frac{\partial \ln n_B}{\partial n_B} = \frac{1}{(1+\alpha)n_B}\left(1 + \alpha - \frac{r\alpha}{1+\alpha}\frac{1-\theta_\alpha^r}{1+\theta_\alpha^r}\right) > 0.$$

The effects of n_A and n_B on conflict intensity $\hat{X}$ are

$$\frac{\partial \hat{X}}{\partial n_A} = \frac{\partial X_A}{\partial n_A} + \frac{\partial X_B}{\partial n_A} > 0,$$
$$\frac{\partial \hat{X}}{\partial n_B} = \frac{\partial X_A}{\partial n_B} + \frac{\partial X_B}{\partial n_B} > 0.$$

A2. Proof of Proposition 2

To show how the change in parameter r affects equilibrium effort levels x_a and x_b, determined by Eq. (2), we consider two cases separately.

First, we consider the trivial case in which $\theta_\alpha = 1$. This occurs if $\theta = 1$ (i.e., $n_A = n_B$) or $\alpha = 0$. Then we get

$$x_a = \left(\frac{Vr}{4n_B^\alpha}\right)^{\frac{1}{1+\alpha}} \text{ and } x_b = \left(\frac{Vr}{4n_A^\alpha}\right)^{\frac{1}{1+\alpha}}.$$

It is straightforward to see that for $r \in (0, 1]$, both x_a and x_b are strictly increasing in r.

Second, we consider the non-trivial case in which $0 < \theta_\alpha < 1$. This occurs if $\alpha > 0$ and $n_A < n_B$. To see the effect of r on x_a and x_b, it suffices to study the effect of r on $G(r)$, where $G(r) \equiv \ln \frac{r\theta_\alpha^r}{(1+\theta_\alpha^r)^2}$. Note that we can rewrite $G(r)$ as

$$G(r) = \ln r + r \ln \theta_\alpha - 2\ln(1+\theta_\alpha^r).$$

For any fixed $V > 0$, $n_A > 0$, $n_B > 0$, and $\alpha > 0$, x_a and x_b are increasing (resp. decreasing) in r if and only if $G(r)$ is increasing (resp. decreasing) in r. To see the effect of r on $G(r)$, we have

$$G'(r) = \frac{1}{r} + \frac{1-\theta_\alpha^r}{1+\theta_\alpha^r}\ln\theta_\alpha \text{ and } G''(r) = -\frac{1}{r^2} - \frac{2\theta_\alpha^r}{(1+\theta_\alpha^r)^2}(\ln\theta_\alpha)^2 < 0,$$

which implies that $G(r)$ is strictly concave. Given that $\lim_{r\to 0+} G'(r) > 0$ and $\lim_{r\to\infty} G'(r) < 0$, there exists a unique $\bar{r} > 0$ such that $G'(\bar{r}) = 0$, i.e., $\bar{r}$ uniquely solves

$$\frac{1}{\bar{r}} + \frac{1-\theta_\alpha^{\bar{r}}}{1+\theta_\alpha^{\bar{r}}}\ln\theta_\alpha = 0.$$

Thus, $G(r)$ is increasing (resp. decreasing) in r if and only if $r < \bar{r}$ (resp. $r > \bar{r}$).

Next we need to compare $\bar{r}$ with 1. Note that $\bar{r}$ must be increasing in θ_α. To see this, let $F(\bar{r}, \theta_\alpha) = \frac{1}{\bar{r}} + \frac{1-\theta_\alpha^{\bar{r}}}{1+\theta_\alpha^{\bar{r}}}\ln\theta_\alpha = 0$. Then we get

$$\frac{\partial F}{\partial \bar{r}} = -\frac{1}{\bar{r}^2} - \frac{2\theta_\alpha^{\bar{r}}}{(1+\theta_\alpha^{\bar{r}})^2}(\ln\theta_\alpha)^2 < 0,$$

$$\frac{\partial F}{\partial \theta_\alpha} = \frac{1-\theta_\alpha^{\bar{r}}}{(1+\theta_\alpha^{\bar{r}})\theta_\alpha} - \frac{2\bar{r}\theta_\alpha^{\bar{r}-1}}{(1+\theta_\alpha^{\bar{r}})^2}\ln\theta_\alpha > 0.$$

By Implicit Function Theorem, we have

$$\frac{d\bar{r}}{d\theta_\alpha} = -\frac{\frac{\partial F}{\partial \theta_\alpha}}{\frac{\partial F}{\partial \bar{r}}} > 0.$$

Moreover, when $\theta_\alpha \to 1$, $\bar{r} \to +\infty$; when $\theta_\alpha \to 0$, $\bar{r} \to 0+$. Therefore, there exists a unique $\bar{\theta}_\alpha \in (0, 1)$ that solves

$$1 + \frac{1-\bar{\theta}_\alpha}{1+\bar{\theta}_\alpha}\ln\bar{\theta}_\alpha = 0.$$

Then we have $\bar{\theta}_\alpha \approx 0.2137$, and $\bar{r} < 1$ if and only if $\theta_\alpha < \bar{\theta}_\alpha$. Therefore, when $\theta_\alpha < \bar{\theta}_\alpha$, x_a and x_b are increasing in r for $r < \bar{r}$, and decreasing in r for $\bar{r} < r < 1$; when $\bar{\theta}_\alpha \leq \theta_\alpha \leq 1$, x_a and x_b are increasing in r for $r \in (0, 1]$.

A3. Proof of result in Section 4.2

In this section, we formally prove that in a bipartite conflict network game, there exists an inverted-U relationship between equilibrium efforts and r, for a general convex cost function, provided that condition (4) holds.

First, we can show that (i) $1 < \frac{X_a}{X_b} < \frac{n_B}{n_A}$; (ii) $\frac{n_A}{n_B} < \frac{x_a}{x_b} < 1$. From Eq. (3) and the definition of X_a and X_b, we get

$$\frac{C'(X_a)X_a}{C'(X_b)X_b} = \frac{n_B}{n_A}.$$

Since $C'(x)x$ is increasing in x, we get $X_b < X_a$, i.e., $\frac{X_a}{X_b} > 1$. Then we also have $\frac{X_a}{X_b} = \frac{C'(X_b)}{C'(X_a)}\frac{n_B}{n_A} < \frac{n_B}{n_A}$. This establishes (i). It is easy to check (ii) holds by the definition of X_a and X_b. By the definition of $g(\cdot)$, we know that $0 < g(x_b) < 1$ for any $x_b > 0$.

Next we show under condition (4), $g'(x_b) \leq 0$. Note that

$$\begin{aligned}
g'(x_b) &= \frac{1}{x_b}\left(\frac{dx_a}{dx_b} - \frac{x_a}{x_b}\right) \\
&= \frac{1}{x_b}\left(\frac{C''(X_b)X_b + C'(X_b)}{C''(X_a)X_a + C'(X_a)} - \frac{x_a}{x_b}\right) \\
&= \frac{1}{x_b^2[C''(X_a)X_a + C'(X_a)]}\left(C''(X_b)X_b x_b - C''(X_a)X_a x_a\right) \\
&= \frac{C'(X_b)}{x_b[C''(X_a)X_a + C'(X_a)]}\left(\frac{C''(X_b)X_b}{C'(X_b)} - \frac{C''(X_a)X_a}{C'(X_a)}\right),
\end{aligned}$$

where the second equality follows from total differentiation of Eq. (3) with respect to x_a and x_b. Then we know $g'(x_b) \leq 0$ if and only if $\frac{C''(X_b)X_b}{C'(X_b)} - \frac{C''(X_a)X_a}{C'(X_a)} \leq 0$. Provided that $X_b < X_a$, condition (4) ensures that $g'(x_b) \leq 0$.

We further define $f(x) = C'(n_A x)x$. Note that $f'(x) > 0$. By the definition of f and g, the equilibrium effort x_b is (uniquely) determined by

$$Vr\frac{g^r(x_b)}{[g^r(x_b)+1]^2} = f(x_b). \tag{5}$$

Note that the uniqueness of x_b follows from the fact that the RHS of Eq. (5) is increasing in x_b, while the LHS is decreasing in x_b, given that $0 < g(x_b) < 1$ and $g'(x_b) \leq 0$.

To show the effect of r on x_b, we take log of Eq. (5) and then totally differentiate it with respect to x_b and r, which gives

$$\frac{dx_b}{dr} = -\frac{\frac{1}{r} + \frac{1-g^r(x_b)}{1+g^r(x_b)}\ln g(x_b)}{\frac{r}{g(x_b)}\frac{1-g^r(x_b)}{1+g^r(x_b)}g'(x_b) - \frac{1}{f(x_b)}f'(x_b)}. \tag{6}$$

Firstly, note that the denominator in Eq. (6) is always negative, given that $f' > 0$, $g' \leq 0$ and $0 < g < 1$. Second, the numerator of Eq. (6) is positive when when r is sufficiently close to zero, and negative when r is sufficiently large, given that $\frac{n_A}{n_B} < g(x_b) < 1$. As a result, we know $\frac{dx_b}{dr} > 0$ when r is sufficiently close to zero, and $\frac{dx_b}{dr} < 0$ when r is large enough otherwise. This establishes the inverted-U relationship between equilibrium efforts and returns to scale technology.

A4. Proof of result in Section 4.3

In a bipartite conflict network game, we need to check the second-order conditions and participation constraints for agents in both coalitions. First, for an agent $i \in A$, define H^a to be an $n_B \times n_B$ matrix where $H^a_{jj} = \frac{\partial^2 u_i}{\partial x_{ij}^2}$ for $j \in B$ and $H^a_{jk} = \frac{\partial^2 u_i}{\partial x_{ij} \partial x_{ik}}$ for distinct j and k in B. In an equilibrium, we can derived that

$$\begin{aligned} \frac{\partial^2 u_i}{\partial x_{ij}^2} &= V \frac{r x_a^{r-2} x_b^r [(r-1)x_b^r - (1+r)x_a^r]}{(x_a^r + x_b^r)^3} - \alpha (n_B x_a)^{\alpha-1} \\ &= -\frac{V r \theta_\alpha^r [(1+r)\theta_\alpha^r - (r-1)]}{x_a^2 (1+\theta_\alpha^r)^3} - \alpha (n_B x_a)^{\alpha-1}, \\ \frac{\partial^2 u_i}{\partial x_{ij} \partial x_{ik}} &= -\alpha (n_B x_a)^{\alpha-1}. \end{aligned}$$

Define $m_{a1} = \frac{V r \theta_\alpha^r [(1+r)\theta_\alpha^r - (r-1)]}{x_a^2 (1+\theta_\alpha^r)^3}$ and $m_{a2} = \alpha (n_B x_a)^{\alpha-1}$. Clearly, $m_{a2} > 0$. Let the k-th order principal minor determinant of H^a be A^a_k. It can be derived that $A^a_k = (-1)^k (m_{a1})^{k-1} (m_{a1} + k m_{a2})$ for $k = 1, 2, \ldots, n_B$. Then H is negative definite if $(-1)^k A^a_k > 0$, i.e., $m_{a1}^{k-1} (m_{a1} + k m_{a2}) > 0$ for $k = 1, 2, \ldots, n_B$. We can show the second-order conditions hold if and only if $m_{a1} > 0$, i.e.,

$$\theta_\alpha^r > \frac{r-1}{1+r}.$$

These condition hold if $r < r_1$, where r_1 is the unique solution to $\psi_1(r) = \theta_\alpha^r - \frac{r-1}{1+r} = 0$. The existence follows from $\psi_1(1) > 0$, $\psi_1(r) < 0$ for $r \to \infty$, and the continuity of ψ_1, while the uniqueness follows from the fact that $\psi_1(r)$ is decreasing in r. Note that we have $r_1 > 1$. Furthermore, we can derive that

$$\frac{\mathrm{d}r_1}{\mathrm{d}\theta_\alpha} = \frac{r_1 \theta_\alpha^{r_1 - 1}}{\frac{2}{(1+r_1)^2} - (\ln \theta_\alpha) \theta_\alpha^{r_1}} > 0,$$

which implies that the threshold r_1 is increasing in θ_α.

Moreover, the participation constraint for each agent is

$$n_B \frac{x_a^r}{x_a^r + x_b^r} V - n_B V - \frac{1}{1+\alpha} (n_B x_a)^{1+\alpha} \geq 0,$$

which requires

$$\frac{r}{1+\theta_\alpha^r} \leq 1 + \alpha.$$

This condition holds if $r \leq r_2$, where r_2 is the unique solution to $\psi_2(r) = \frac{r}{1+\theta_\alpha^r} - (1+\alpha) = 0$. The existence follows from $\psi_2(1) < 0$, $\psi_2(r) > 0$ for $r \to \infty$, and the continuity of ψ_2, while the uniqueness follows from that $\psi_2(r)$ is increasing in r. We also get that $r_2 > 1$. Furthermore, we can derive that

$$\frac{\mathrm{d}r_2}{\mathrm{d}\theta_\alpha} = \frac{r_2^2 \theta_\alpha^{r_2 - 1}}{1 + \theta_\alpha^{r_2} - r_2 \theta_\alpha^{r_2} (\ln \theta_\alpha)} > 0,$$

which implies that the threshold r_2 is also increasing in θ_α.

Following a similar argument, for an agent $j \in B$, the second-order conditions are satisfied if and only if

$$(1+r) - \theta_\alpha^r (r-1) > 0,$$

which always holds. The participation constraint is

$$n_A \frac{x_b^r}{x_a^r + x_b^r} V - \frac{1}{1+\alpha} (n_A x_b)^{1+\alpha} \geq 0,$$

which requires

$$\frac{r \theta_\alpha^r}{1+\theta_\alpha^r} \leq 1 + \alpha,$$

which is also satisfied given $r \leq r_2$.

In sum, a set of necessary conditions that ensure the existence of equilibrium are $r < r_1$ and $r \leq r_2$. Since both r_1 and r_2 are increasing in θ_α, when θ_α is sufficiently small (i.e., the group size difference is sufficiently large and/or the cost function is sufficiently convex), r cannot be too large in order that the equilibrium exists. In particular, when θ_α is sufficiently close to zero, the threshold r_1 would also be very close to 1. This implies that the equilibrium is less likely to arise when $r > 1$.

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