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Competition for networked agents in the lottery Blotto game

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ABSTRACT

This paper examines a duopoly setting in which two firms target their marketing budgets to agents embedded in a social network. Depicting the competition between two firms as a lottery Blotto game, we characterize the equilibrium marketing strategies and study how the network externality affects firms' marketing decision. Examples of networks are further provided to illustrate how the marketing strategies depend on the agents' network structures and the strength of network effect.

1. Introduction

It is widely acknowledged that agents are embedded in a social network, they make consumption decisions based on whether their close friends, neighbors, and celebrities also adopt the same products. Adolescents' consumption of alcohol and tobacco is affected by their friends' consumption. When deciding whether to use a new web conference platform, enterprise staffs rely on information from colleagues and peers. Such peer influence can also be found in school choice, job searching and criminal offense.[1]

In the meanwhile, firms in the market make marketing effort to attract the attention from agents, increase the consumption of its products. Consider, for example, in the public cloud service market, enterprises can choose among AWS, Microsoft Azure and Google Cloud Platform. In the market of social media messaging apps, Facebook Messenger, Snapchat, and WhatsApp are fighting for market shares. Considering agents are more inclined to choose products that are used by more peers, service providers need to make an effort to lobby "important" agents and strive to induce agents to choose their own products among similar products.

In this paper, we study marketing competition between firms when products are differentiated and exhibit local (positive) network effects. Two products, provided by two firms, are perfect substitutable and agents can freely access to them. As in Chen et al. (2018b), we assume that an agent's utility consists of two parts: one part corresponds to the utility from her own usage level, and the other part describes the positive externality of her peers.[2] In contrast, to capture the competition between firms, we assume that each agent's total consumption level for the two products will not be affected by firms' marketing effort.[3] The duopoly firms make efforts such as advertising to target their marketing budgets to agents in the market.

How can the duopoly firms exploit the above network externality, and allocate their marketing budgets to agents so as to maximize the total consumption towards their products? To answer this question, we model the competition between firms as a lottery Blotto game adopted from Xu and Zhou (2018). Viewing each consumer's consumption level as a prize in the competition between firms, we characterize the equilibrium marketing strategies and study how the strength of network effect and network structure affect firms' budget allocation decision.

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1 See Chen et al. (2018a) for a detailed review.

2 Using linear–quadratic utility functions to study network games with strategic complementarities among players is firstly provided by Ballester et al. (2006).

3 This assumption is reasonable since in some industry such as tobacco, the impact of advertising on consumption is negligible, or at least very minor compared to other social and cultural factors.

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We show that the equilibrium consumption level for each agent is the agent's Bonacich centrality. The fraction of a firm's marketing budget targeted to an individual is just the proportion of individual consumption in total market consumption. In other words, firms' equilibrium marketing strategies only depend on the underlying social network structure and the strength of network effect. Specifically, when the strength of network effect is relatively small, the network externality mainly comes from the number of neighbors. While when the strength of network effect from peers is large enough, firms' tend to allocate more budget to those agents with higher eigenvector centralities, who are more influential.

This paper belongs to the literature on firms' competition in social networks. The Bonacich centrality measure always appears in social and economic networks literature. Following the seminal work of Ballester et al. (2006), a bunch of papers expand research on the above topics.[4] The social network-based pricing decision has been well studied by Candogan et al. (2012), Bloch and Querou (2013), Fainmesser and Galeotti (2016) and many others. While early papers primarily focus on the optimal pricing strategies of a monopolist, Chen et al. (2018b) study the price competition between competing firms who sell heterogeneous products. They show that firms' discriminatory pricing based on network structure is related to the Bonacich centrality measures. While all of the literature assumes that consumers buy product from a given manufacturer, our paper differs from these by allowing agents to choose which firm to buy from, and we study competition between firms on agents' consumption while pricing is no longer firms' strategy in our model.

Our study is related to the literature on social network-based marketing decision, including Hartline et al. (2008), Carroni et al. (2020), and Manshadi et al. (2020).[5] While these studies focus on the optimal marketing or seeding strategies of a monopolist, competition between firms on targeted advertising has received relatively little attention. Using the Blotto game (e.g., Friedman, 1958; Roberson, 2011), Bimpikis et al. (2016) study consumers' awareness levels for firms in the word-of-mouth process and highlight their dependence on the underlying social network structure. Two firms make advertising efforts to compete for agents' awareness according to contest success functions. Goyal et al. (2019) consider a stochastic dynamics of local adoption model. Two firms choose an allocation of budget to "seed" the initial adoption of their products, so as to maximize the total number of eventual product adoptions. They use the linear selection function to specify the probability of infection by each firm in terms of the local relative market share split. Similar with these two papers, our paper also uses contest success functions to model the competition between firms with marketing budget. While differ from the literature considering the dynamic process on social networks, we assume that when firms make marketing efforts to compete for agents' consumptions, firms' marketing strategies will not affect the amount of consumption of each agent. Our main focus is to study the optimal budget allocation of firms and how network externality influences it.

[4] Ballester et al. (2006) are the first to show that the Nash equilibrium in effort is proportional to the Bonacich centrality of each agent. The Bonacich centrality measure always appears in social and economic networks literature. See Jackson et al. (2017) for a detailed literature review.

[5] Without using the Bonacich centrality measure, these three papers discuss the role of social networks in implementing marketing strategies for a monopolist. Hartline et al. (2008) use a general influence model to identify a family of marketing strategies called influence-and-exploit strategies. Taking Word-of-Mouth (WoM) as a powerful marketing force, Manshadi et al. (2020) study the impact of heterogeneity in the degree of connections on the cost and speed of diffusion as well as on optimal seeding strategies. Carroni et al. (2020) classify the Word-of-Mouth communication channels into two categories: Private WoM and Public WoM. The use of different channels implies different network structures, and they explore the reward structure to generate the optimal levels of communication channels.

2. Model setup

Consider a market with a set $\mathcal{N} = \{1, 2, \dots, N\}$ of agents (consumers) and two firms (producers). Agents are embedded in a connected social network represented by an $N \times N$ matrix $\mathbf{G} = (g_{ij})_{N\times N}$. We assume $g_{ij} = 1$ if i and j are connected for all i, j, otherwise $g_{ij} = 0$. Each agent cannot connect with himself, i.e., $g_{ii} = 0$. The network structure is symmetric in the sense that $g_{ij} = g_{ji}$. Let N_i denote the set of agent i's neighbors: $N_i = \{j \in \mathcal{N} : g_{ij} = 1\}$. Let $d_i = |N_i|$ denote the number of agent i's neighbors.

Assume the prices of the consumption goods are fixed, then agent i's utility function can be expressed as follows[6]

$$U_i(a_i, \mathbf{a}_{-i}) = a_i - \frac{1}{2}a_i^2 + \delta \sum_{j\neq i} g_{ij} a_i a_j, \tag{1}$$

where $a_i \in \mathbf{R}_+$ is the amount of consumption that agent i chooses, and $\mathbf{a}_{-i}$ denotes the consumption levels of all the other agents. Let $\mathbf{a} = (a_1, a_2, \dots, a_N)$ denote the consumption levels of N agents. The first two terms in Eq. (1) represent the utility that agent i derives from her own consuming a_i. The quadratic form of the utility function, which captures the decreasing marginal returns from consumption, allows for tractable analysis. The third term with $\delta > 0$ represents the positive network externality. A higher δ means a higher strength of network effect, i.e., agent i's utility depends more on the consumption of her peers.

Two firms A and B make marketing efforts to compete for consumption from agents. The network matrix $\mathbf{G}$ is known to the firms. Suppose that each firm has a marketing *budget* $T_F \in \mathbf{R}_+$; each firm F allocates the budget across the N agents, $\mathbf{x}_F = (x_{F1}, x_{F2}, \dots, x_{FN})$, where $x_{Fi} \geq 0$ and $\sum_{i\in\mathcal{N}} x_{Fi} \leq T_F$. We further assume firms' marketing behaviors (x_{Ai}, x_{Bi}) to agent i do not affect her consumption level a_i. The competition on each agent i's consumption is modeled as a contest: if firm F wins, then it obtains an amount a_i of consumption from agent i. The outcome of each contest depends on the simultaneously strategic behaviors (i.e., (x_{Ai}, x_{Bi})) of firms, which is specified by a contest success function following Tullock (1980). Specifically, in the contest for agent i's consumption, the winning probability of firm F is

$$p_{Fi}(x_{Ai}, x_{Bi}) = \begin{cases} \frac{x_{Fi}^r}{x_{Ai}^r + x_{Bi}^r}, & \text{if } x_{Ai} + x_{Bi} > 0, \\ \frac{1}{2}, & \text{if } x_{Ai} = x_{Bi} = 0, \end{cases}$$

where r is a positive constant. The parameter r represents the *returns to scale technology in marketing effort*. Throughout the paper, we assume the returns to scale are constant or decreasing ($r \leq 1$), which ensures the existence and uniqueness of a pure-strategy Nash equilibrium in our model.

The competition between two firms can be viewed as an N-battle lottery Blotto game. In battle i, firms compete for agent i's consumption. Given the prices of the consumption goods, each firm chooses the marketing strategy x_F to maximize the total consumption from all agents, i.e.,

$$\max_{\mathbf{x}_F} \pi_F(\mathbf{x}_A, \mathbf{x}_B; \mathbf{a}) = \sum_{i\in\mathcal{N}} \frac{x_{Fi}^r}{x_{Ai}^r + x_{Bi}^r} a_i,$$

subject to the budget constraint. Firms and agents make their decisions simultaneously.

[6] Differ from the literature studying firms' pricing strategies in social networks, our focus is firms' marketing strategies. Therefore, we assume the prices of the consumption goods are fixed. Then the coefficient of the first term in agent's utility function can be normalized to one. The robustness of our results with relaxing this assumption will be discussed in Section 5.

3. Equilibrium analysis

Let $\mathbf{I}$ denote the n-dimensional identity matrix, and $\mathbf{1}$ denote the n-dimensional column vector of ones. Note that $\mathbf{G}$ is a real symmetric matrix, Spectral Theorem implies that each eigenvalue of $\mathbf{G}$ is real. We assume that the distinct eigenvalues of $\mathbf{G}$ are $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ ($k \leq N$), then the largest eigenvalues of $\mathbf{G}$ is λ_1. Let $\mathbf{b}(\mathbf{G}, \delta, \mathbf{1}) = (\mathbf{I} - \delta\mathbf{G})^{-1}\mathbf{1}$ denote the vector of Bonacich centralities of parameter δ.[7] Based on , we first present agents' equilibrium consumption levels in the social network.

Lemma 1 (*Ballester, Calvo-Armengol and Zenou, 2006*)**.** *If $\delta < \frac{1}{\lambda_1}$, then the unique equilibrium of the consumption game takes the following form $\mathbf{a}^* = \mathbf{b}(\mathbf{G}, \delta, \mathbf{1})$.*

Note that b_i is the ith entry of vector $\mathbf{b}(\mathbf{G}, \delta, \mathbf{1})$. Lemma 1 shows that the equilibrium consumption level for each agent i is just its Bonacich centrality, i.e., $a_i^* = b_i$.

Given each agent's consumption level, now we are able to derive the equilibrium marketing level for each firm. In each battle i, two firms compete for consumption a_i from agent i, therefore a_i can be regarded as a common "prize" in the contest. Friedman (1958) considers the budget allocation model with $r = 1$, and Xu and Zhou (2018) derives an unique equilibrium under a general setting with heterogeneous returns to scale technologies in each battle. Based on their results, we can obtain the following solution.

Lemma 2. *Each firm F's equilibrium marketing strategy to agent i is $x_{Fi}^* = \frac{b_i}{\sum_{j\in\mathcal{N}} b_j} T_F$, $F = A, B$.*

Lemma 2 shows that in the competition on agent i's consumption, each firm's equilibrium marketing level x_{Fi}^* is proportional to its initial budget. The coefficient $\frac{b_i}{\sum_{j\in\mathcal{N}} b_j}$ is just the proportion of agent i's consumption b_i in total market consumption $\sum_{j\in\mathcal{N}} b_j$, which only depends on the Bonacich centrality of each agent. Agents with higher Bonacich centralities are viewed as more *important* agents. Firms would like to make more marketing effort to those agents with *higher* consumption levels. And we have the following corollary immediately.

Corollary 1. *(i) The ratio of a firm's equilibrium marketing level to two agents is $\frac{x_{Fi}^*}{x_{Fj}^*} = \frac{b_i}{b_j}$.*

(ii) The ratio of two firms' equilibrium marketing levels to agent i is $\frac{x_{Ai}^*}{x_{Bi}^*} = \frac{T_A}{T_B}$.

(iii) Firms' equilibrium marketing strategies are irrelevant to the returns to scale technology r.

Corollary 1 implies that the ratio of each firm's equilibrium marketing level to two agents is just the ratio of the Bonacich centralities. To each agent, the ratio of two firms' equilibrium marketing levels is just the ratio of two firms' budgets, which is unrelated to the Bonacich centrality. Moreover, the returns to scale technology in marketing effort will not affect firms' equilibrium marketing strategies.[8]

Based on the above equilibrium results, we will consider the impact of network effect (δ) on firms' marketing levels. We first define eigenvector centrality: Eigenvector centrality is computed by assuming that the centrality of node i is proportional to the sum of centrality of node i's neighbors $\lambda e_i = \sum_{j\in\mathcal{N}} g_{ij} e_j$, where λ is a positive proportionality factor. In matrix terms, $\lambda\mathbf{e} = \mathbf{G}\mathbf{e}$.[9] Recall that λ_1 is the largest eigenvalue of $\mathbf{G}$. Denote by $\mathbf{e} = (e_1, e_2, \ldots, e_N)^T$ the corresponding eigenvector of λ_1. Then e_i is the ith entry of $\mathbf{e}$, called eigenvector centrality of agent i.

Proposition 1. *In the competition for agent i's consumption:*

(i) When $\delta = 0$, the equilibrium marketing level for each firm F is $x_{Fi}^* = \frac{1}{N} T_F$.

(ii) When $\delta \approx 0$, the equilibrium marketing level for each firm F is $x_{Fi}^* \approx \frac{1+\delta d_i}{N+\delta\sum_{j\in\mathcal{N}} d_j} T_F$.

(iii) When $\delta \uparrow \frac{1}{\lambda_1}$, the ratio of a firm's equilibrium marketing level to two agents $\frac{x_{Fi}^*}{x_{Fj}^*} \to \frac{e_i}{e_j}$.

Proof. The proofs of (i) and (ii) are simple and straightforward. For (iii), based on Perron–Frobenius Theorem, the matrix $\mathbf{G}$ has a largest eigenvalue $\lambda_1 > 0$ (with algebraic multiplicity 1) with a positive eigenvector $\mathbf{e} = (e_1, e_2, \ldots, e_N)^T$. We can get a diagonalization for $\mathbf{G}$, and a simplified expression of $(\mathbf{I} - \delta\mathbf{G})^{-1}$. Since $\mathbf{b} = (\mathbf{I} - \delta\mathbf{G})^{-1}\mathbf{1}$, by a series of calculations, the ith component b_i can be expressed as (3). When δ approaches the upper bound $\frac{1}{\lambda_1}$, b_i goes to infinite, while the ratio $\frac{b_i}{b_j}$ approaches to $\frac{e_i}{e_j}$. Note that b_i solves the linear system $b_i = 1 + \delta\sum_{j\neq i} g_{ij} b_j$, when $b_i \to +\infty$, this equation can be reduced to $e_i = \frac{1}{\lambda_1}\sum_{j\neq i} g_{ij} e_j$, or $\mathbf{G}\mathbf{e} = \lambda_1\mathbf{e}$, which is the equation for the first eigenvector. The details are relegated to Appendix. ■

Proposition 1(i) shows that without network effect, each agent's utility only depends on his own consumption, which is one. Therefore, a firm's equilibrium marketing level for each agent is the same.

Proposition 1(ii) shows that when the effect from peers is relatively small, the network externality mainly comes from the number of neighbors (d_i). When firms make marketing decisions, only *local* effect in the social network need to be taken into account. Agents with more neighbors will consume more, and firms will make more marketing effort to these agents.

While when the effect from peers is large enough, i.e., $\delta \uparrow \frac{1}{\lambda_1}$, each agent's utility not only depends on the number of her peers, but also depends on peers' consumption levels. Since agents' consumption behaviors are complementary, an increase of network effect will enhance the mutual influence among agents, thus forming a *global* effect. Therefore, the equilibrium consumption level for each agent goes to infinite. Each agent's *influence*, which is measured by eigenvector centrality, plays a crucial rule in determining firms' marketing strategies. Proposition 1(iii) shows that the ratio of a firm's equilibrium marketing level to two agents will converge to the ratio of their eigenvector centralities. Firms will conduct marketing activities according to the influence of agents, and devote more resources to win over influential agents.

4. Examples

To better understand the effect of network structure on the consumption of agents and the budget allocation of firms, we provide three examples.

[7] Bonacich centrality, a measure of prestige or centrality based on the number of walks emanating from a node, is proposed by Bonacich (1987).

[8] Jiao et al. (2019) find an inverted U relationship between the returns to scale technology and (individual and total) equilibrium efforts in a complete bipartite conflict network.

[9] See Jackson (2008).

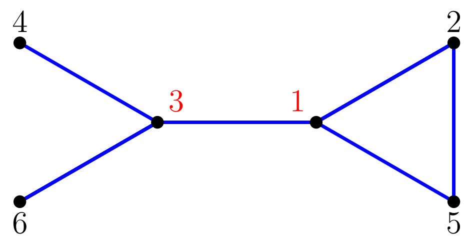

Fig. 1. An asymmetric network with six agents.

Example 1 (*A Regular Network*)**.** A network G is regular of degree d if each agent has exactly d neighbors. Using Lemma 1, we obtain the following equilibrium consumptions:

$$\mathbf{a}^* = \mathbf{b} = \frac{1}{1-d\delta}\mathbf{1}.$$

From Lemma 2, the equilibrium allocation for firm F in battle i is given by

$$x_{Fi}^* = \frac{1}{N}T_F, \quad F = A, B.$$

Given the regular network structure, the consumption level of each agent is the same. Thus, firms distribute marketing budget equally to all agents.

Example 2 (*A Star Network*)**.** Agent 1 is located in the center, and other five agents are located at the periphery. Using Lemma 1, we obtain the following equilibrium consumptions:

$$\mathbf{a}^* = \mathbf{b} = \left(\frac{1+5\delta}{1-5\delta^2}, \frac{1+\delta}{1-5\delta^2}, \frac{1+\delta}{1-5\delta^2}, \frac{1+\delta}{1-5\delta^2}, \frac{1+\delta}{1-5\delta^2}, \frac{1+\delta}{1-5\delta^2}\right)^T.$$

Lemma 2 provides the equilibrium allocation of firm F for agents in the star network:

$$x_{F1}^* = \frac{1+5\delta}{6+10\delta}T_F, x_{F2}^* = \cdots = x_{F6}^* = \frac{1+\delta}{6+10\delta}T_F, \quad F = A, B.$$

Note that the agent located in the center has better access to the market than the peripheral agents. Since $\frac{\partial x_{F1}^*}{\partial \delta} = \frac{5}{(3+5\delta)^2}T_F > 0$ and $\frac{\partial x_{F2}^*}{\partial \delta} = \cdots = \frac{\partial x_{F6}^*}{\partial \delta} = -\frac{1}{(3+5\delta)^2}T_F < 0$, an increase in network effect δ makes the center agent more important, which leads more budget of firms allocated to the center and less budget to the periphery.

Example 3 (*An Asymmetric Network*)**.** As shown in Fig. 1, both agents 1 and 3 have three neighbors, both agents 2 and 5 have two neighbors, and both agents 4 and 6 have only one common neighbor.

Then the network is represented by the matrix

$$\mathbf{G} = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Using Lemma 1, we have the equilibrium consumption as follows in Box I. Firm F allocates its budget among the agents $(x_{F1}^*, x_{F2}^*, x_{F3}^*, x_{F4}^*, x_{F5}^*, x_{F6}^*)$ is given in Box II.

The largest eigenvalue of G is $\lambda_1 = 2.2784$, and the corresponding eigenvector of λ_1 is $\mathbf{e} = (0.2561, 0.2003, 0.1828, 0.0803, 0.2003, 0.0803)^T$.

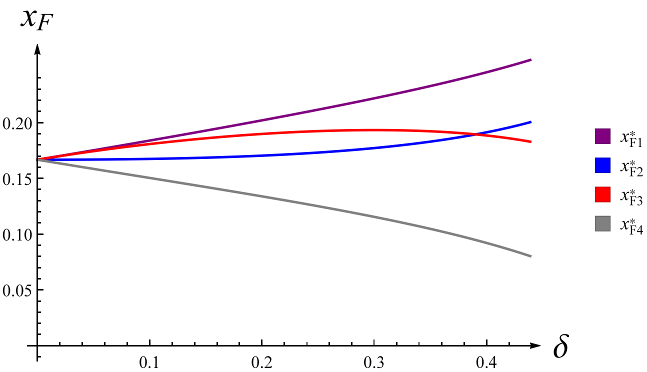

Fig. 2. Budget allocations of firms.

For simplicity, set $T_F = 1, F = A, B$. In Fig. 2, we plot a firm's budget allocations to agents 1,2,3,4 as a function of δ, respectively.[10]

Given the number of agents, Examples 1–3 show that the network structures will affect agents' behavior, and thus the budget allocation of firms.

In the meanwhile, Example 3 highlights a few observations about the network effect of agents. Firstly, x_{F1}^* is strictly larger than x_{F2}^* and both of them are increasing in δ. This is because agent 1 has more connections comparing with agent 2 and thus on a more central position. Secondly, x_{F4}^* is decreasing in δ since agent 4 is connected with only one peer and on the least central position. As δ increases in magnitude, the budget allocation to the agents with least influence will decrease. Thirdly and most importantly, x_{F3}^* increases first and then decreases as δ increases. This can be explained by Proposition 1: When δ is small, the network externality mainly comes from the number of connected peers, therefore firms will allocate more budget in agent 3, who has three connected peers. While when δ becomes large enough, the influence of the connected peers plays a more important rule in firms' marketing decisions. To agent 3, the network effect from least influential agents 4 and 6 dominates that from agent 1 with most influence. Last but not least, although agent 3 has more neighbors than agent 2, a sufficiently large network effect $\left(\delta \uparrow \frac{1}{\lambda_1} = 0.4389\right)$ makes firm F allocate more budget in agent 2 since agent 2 is more influential ($e_2 > e_3$). Example 3 also confirms our main result in Proposition 1(iii), that is $\frac{x_{Fi}^*}{x_{Fj}^*} \to \frac{e_i}{e_j}$ when $\delta \uparrow \frac{1}{\lambda_1}$.

5. Conclusion

In this paper, we consider firms' competition for attention of networked agents. Two firms, targeting their marketing budgets to individuals embedded in a social network, compete for agents' consumption. The competition between firms is modeled as a lottery Blotto game. We characterize the equilibrium marketing strategies and highlight their dependence on the underlying network externality.

Our main focus of this paper is to study the effect of network externality on each agent's consumption and firms' marketing effort. We assume prices of the consumption goods for each agent are fixed and demand is inelastic. If the market prices are still taken as given for both firms and agents, then the change of

10 Due to the symmetry of the network structure, we have $x_{F2}^* = x_{F5}^*$ and $x_{F4}^* = x_{F6}^*$.

$$\mathbf{a}^* = \mathbf{b} = \frac{(1+2\delta-\delta^2-4\delta^3, 1+\delta-2\delta^2, 1+2\delta-3\delta^2-4\delta^3, 1-3\delta^2, 1+\delta-2\delta^2, 1-3\delta^2)^T}{1-\delta-5\delta^2+3\delta^3+4\delta^4}.$$

Box I.

$$(x^*_{F1}, x^*_{F2}, x^*_{F3}, x^*_{F4}, x^*_{F5}, x^*_{F6}) = \frac{(1+2\delta-\delta^2-4\delta^3, 1+\delta-2\delta^2, 1+2\delta-3\delta^2-4\delta^3, 1-3\delta^2, 1+\delta-2\delta^2, 1-3\delta^2)}{6+6\delta-14\delta^2-8\delta^3}T_F.$$

Box II.

price only affects the coefficient of the first term in agent's utility function. The equilibrium consumption levels and marketing efforts will be adjusted accordingly, while the results in Proposition 1(iii) still hold since the equilibrium marketing strategies do not depend on the above coefficient. However, if firms can simultaneously choose marketing strategies and pricing strategies, the interaction between consumers and firms remains open. Candogan et al. (2012) characterize optimal pricing strategies of a monopolist, and show that if the monopolist can perfectly price discriminate the agents, the optimal prices do not depend on the network structure when the interaction matrix is symmetric. Following their argument, if two firms in our model are allowed to collude in setting an individual price for each of the agents, the agents' equilibrium consumption levels still only depend on Bonacich centrality measure, therefore the key insights in our paper are still valid.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix

Proof of Proposition 1. (i) When $\delta = 0$, there is no network effect, then $\mathbf{a}^* = \mathbf{1}$. According to Lemma 2, $x^*_{Fi} = \frac{1}{N}T_F$, for $F = A, B$.

(ii) When δ is small, using Taylor expansion, we have $b_i = 1 + \delta d_i + o(\delta^2)$. According to Lemma 2, $x^*_{Fi} \approx \frac{1+\delta d_i}{\sum_{j\in\mathcal{N}}(1+\delta d_j)}T_F = \frac{1+\delta d_i}{N+\delta\sum_{j\in\mathcal{N}} d_j}T_F$, for $F = A, B$.

(iii) We assume the algebraic multiplicity of each λ_i is t_i. Note that $\sum_{m=1}^k t_m = N$. Spectral Theorem implies that (1) $|\lambda_1| > |\lambda_j|$ for $j \neq 1$; (2) each eigenvalue's geometric multiplicity is equal to its algebraic multiplicity t_i; (3) for each eigenvalue λ_i, there is an orthogonal basis $\{\eta_i^1, \ldots, \eta_i^{t_i}\}$ for its eigenspace.

Since the network is connected, Perron–Frobenius Theorem implies that (1) the largest eigenvalue λ_1 is positive; (2) λ_1's algebraic multiplicity t_1 is 1; (3) there is a (column) eigenvector $\mathbf{e} = (e_1, e_2, \ldots, e_N)^T$ of λ_1 such that the length of $\mathbf{e}$ is 1 and each component e_i is positive.

Let $\mathbf{H} = (e, \underbrace{\eta_2^1, \ldots, \eta_2^{t_2}}_{\text{a basis for } \lambda_2}, \underbrace{\eta_3^1, \ldots, \eta_3^{t_3}}_{\text{a basis for } \lambda_3}, \ldots, \underbrace{\eta_k^1, \ldots, \eta_k^{t_k}}_{\text{a basis for } \lambda_k})$. Then $\mathbf{H}$ is an $N \times N$ orthogonal matrix and $\mathbf{H}^{-1} = \mathbf{H}^T$. Moreover, we have

$$\mathbf{H}^{-1}\mathbf{G}\mathbf{H} = \begin{pmatrix} \lambda_1 & & & & \\ & \lambda_2\mathbf{I}_{t_2} & & & \\ & & \lambda_3\mathbf{I}_{t_3} & & \\ & & & \ddots & \\ & & & & \lambda_k\mathbf{I}_{t_k} \end{pmatrix},$$

which is denoted by $\boldsymbol{\Lambda}$, where each $\mathbf{I}_{t_j}$ denotes the t_j-dimensional identity matrix.

It follows that

$$(\mathbf{I}-\delta\mathbf{G})^{-1} = (\mathbf{H}\mathbf{H}^{-1} - \delta\mathbf{H}\boldsymbol{\Lambda}\mathbf{H}^{-1})^{-1} = (\mathbf{H}(\mathbf{I}-\delta\boldsymbol{\Lambda})\mathbf{H}^{-1})^{-1} = \mathbf{H}(\mathbf{I}-\delta\boldsymbol{\Lambda})^{-1}\mathbf{H}^{-1}. \quad (2)$$

By plugging the expressions of $\mathbf{H}$, $\mathbf{H}^{-1}$ and $\boldsymbol{\Lambda}$ into Eq. (2), we have

$$(\mathbf{I}-\delta\mathbf{G})^{-1} = \left(e, \eta_2^1, \ldots, \eta_2^{t_2}, \ldots, \eta_k^1, \ldots, \eta_k^{t_k}\right) \times \begin{pmatrix} 1-\delta\lambda_1 & & & \\ & (1-\delta\lambda_2)\mathbf{I}_{t_2} & & \\ & & \ddots & \\ & & & (1-\delta\lambda_k)\mathbf{I}_{t_k} \end{pmatrix}^{-1} \begin{pmatrix} e^T \\ (\eta_2^1)^T \\ \vdots \\ (\eta_2^{t_2})^T \\ \vdots \\ (\eta_k^1)^T \\ \vdots \\ (\eta_k^{t_k})^T \end{pmatrix}$$

$$= \left(\frac{1}{1-\delta\lambda_1}e, \frac{1}{1-\delta\lambda_2}\eta_2^1, \ldots, \frac{1}{1-\delta\lambda_2}\eta_2^{t_2}, \ldots,\right.$$

$$\mathbf{H}_1 = \begin{pmatrix} \frac{1}{1-\delta\lambda_1}\zeta_{11} & \frac{1}{1-\delta\lambda_2}\zeta_{21} & \cdots & \frac{1}{1-\delta\lambda_2}\zeta_{t_1+t_2,1} & \cdots & \frac{1}{1-\delta\lambda_k}\zeta_{t_1+\cdots+t_{k-1}+1,1} & \cdots & \frac{1}{1-\delta\lambda_k}\zeta_{N1} \\ \frac{1}{1-\delta\lambda_1}\zeta_{12} & \frac{1}{1-\delta\lambda_2}\zeta_{22} & \cdots & \frac{1}{1-\delta\lambda_2}\zeta_{t_1+t_2,2} & \cdots & \frac{1}{1-\delta\lambda_k}\zeta_{t_1+\cdots+t_{k-1}+1,2} & \cdots & \frac{1}{1-\delta\lambda_k}\zeta_{N2} \\ \vdots & \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ \frac{1}{1-\delta\lambda_1}\zeta_{1N} & \frac{1}{1-\delta\lambda_2}\zeta_{2N} & \cdots & \frac{1}{1-\delta\lambda_2}\zeta_{t_1+t_2,N} & \cdots & \frac{1}{1-\delta\lambda_k}\zeta_{t_1+\cdots+t_{k-1}+1,N} & \cdots & \frac{1}{1-\delta\lambda_k}\zeta_{NN} \end{pmatrix},$$

$$\mathbf{H}_2 = \begin{pmatrix} \zeta_{11} & \zeta_{12} & \cdots & \zeta_{1N} \\ \zeta_{21} & \zeta_{22} & \cdots & \zeta_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \zeta_{N1} & \zeta_{N2} & \cdots & \zeta_{NN} \end{pmatrix}.$$

Box III.

$$\left.\frac{1}{1-\delta\lambda_k}\eta_k^1, \ldots, \frac{1}{1-\delta\lambda_k}\eta_k^{t_k}\right) \begin{pmatrix} e^T \\ (\eta_2^1)^T \\ \vdots \\ (\eta_2^{t_2})^T \\ \vdots \\ (\eta_k^1)^T \\ \vdots \\ (\eta_k^{t_k})^T \end{pmatrix}.$$

For notational simplicity, we relabel the columns of $\mathbf{H}$ as follows: e is relabeled as ζ_1, and each η_i^m ($i = 2, 3, \ldots, k$ and $m = 1, 2, \ldots, t_i$) is relabeled as $\zeta_{\sum_{\ell=1}^{i-1} t_\ell + m}$. Thus, $\mathbf{H}$ is rewritten as $(\zeta_1, \zeta_2, \ldots, \zeta_N)$, where each $\zeta_i = (\zeta_{i1}, \zeta_{i2}, \ldots, \zeta_{iN})^T$ is the ith column. Then the above expression becomes $\mathbf{H}_1\mathbf{H}_2$, where $\mathbf{H}_1$ and $\mathbf{H}_2$ are given in Box III.

Since $\mathbf{b} = (\mathbf{I} - \delta\mathbf{G})^{-1}\mathbf{1}$, the ith component b_i is the summation of all the entries in the ith row of $\mathbf{H}_1\mathbf{H}_2$. That is,

$$b_i = \frac{\zeta_{1i}}{1-\delta\lambda_1}\sum_{\ell=1}^N \zeta_{\ell 1} + \frac{\zeta_{2i}}{1-\delta\lambda_2}\sum_{\ell=1}^N \zeta_{\ell 2} + \cdots + \frac{\zeta_{t_1+t_2,i}}{1-\delta\lambda_2}\sum_{\ell=1}^N \zeta_{\ell,t_1+t_2} + \cdots + \frac{\zeta_{t_1+\cdots+t_{k-1}+1,i}}{1-\delta\lambda_k}\sum_{\ell=1}^N \zeta_{\ell,t_1+\cdots+t_{k-1}+1} + \cdots + \frac{\zeta_{Ni}}{1-\delta\lambda_k}\sum_{\ell=1}^N \zeta_{\ell N}. \quad (3)$$

Since $|\lambda_1| > |\lambda_j|$ for $j \neq 1$, $\lim_{\delta\uparrow\frac{1}{\lambda_1}} b_i = +\infty$. Moreover, we have

$$\lim_{\delta\uparrow\frac{1}{\lambda_1}} \frac{x_{Fi}^*}{x_{Fj}^*} = \lim_{\delta\uparrow\frac{1}{\lambda_1}} \frac{b_i}{b_j} = \frac{\frac{\zeta_{1i}}{1-\delta\lambda_1}\sum_{\ell=1}^N \zeta_{\ell 1}}{\frac{\zeta_{1j}}{1-\delta\lambda_1}\sum_{\ell=1}^N \zeta_{\ell 1}} = \frac{\zeta_{1i}}{\zeta_{1j}} = \frac{e_i}{e_j}. \quad \blacksquare$$

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